Jacobs University
School of Engineering and Science

Fall Term 2011

Introductory Complex Analysis, Homework 7

Due Date: Friday, December 2, in class.

Problems marked ($\star$) are bonus ones.

7.1. Show that for any real $t > 1$ the equation

$$te^{t-z} = 1$$

has a unique solution z_0 in the unit disk. Show that $z_0 \in \mathbb{R}$.

7.2. Fix $k \in (0, 1)$.

(a) Show that the map

$$w(z) = \int_0^z \frac{dt}{\sqrt{1-t^2}\sqrt{1-k^2t^2}}$$

maps conformally the upper halfplane onto a rectangle.

(b) Draw the rectangle by using the notation

$$K = \int_0^1 \frac{dt}{\sqrt{1-t^2}\sqrt{1-k^2t^2}}, \qquad K' = \int_1^{1/k} \frac{dt}{\sqrt{1-t^2}\sqrt{1-k^2t^2}}$$

(c) Show that the inverse function $z(w)$ extends to a meromorphic function $h(w)$ on $\mathbb{C}$.

(d) Show that $h(w)$ has periods $4K$ and $2iK'$.

7.3. Find explicitely a conformal map $g(z)$ of the unit disk onto the vertical half-strip $\mathrm{Im} w > 0$, $-a < \mathrm{Re} w < a$.

7.4. Draw diagrams of Riemann surfaces of the following functions:

(a) $\sqrt{z} + \sqrt{z-1}$; (b) $\sqrt{z} + \sqrt[3]{z}$; (c) $\sqrt[4]{z-1} + \sqrt{z^2-1}$; (d) $\sqrt[4]{z-1}\sqrt{z}$; (e) $\dfrac{\sqrt{z^1-1}}{\sqrt[4]{z+1}}$;

(f) $\sqrt{1-\sqrt{z-1}}$; (g) $\sqrt[3]{\sqrt{z}-2}$.

7.5. ($\star$) Let $f(z) = \sum c_n z^n$ in the unit disk, and all c_n are real non-negative. Show that the point $z = 1$ is singular.