Jacobs University
School of Engineering and Science

Fall Term 2011

Linear Algebra I, Homework 1

Due Date: Friday, September 16, in class.

Problems marked ($\star$) are bonus ones.

1.1. Let F be an field, $a, b, c, d \in F$. Show that

(a) $((a+b)+c)+d = a+(b+(c+d))$;

(b) $-(a+b) = (-a)+(-b)$;

(c) the equations $a+x=b$ and $ax=b$ have a unique solution in F (for $a \neq 0$);

(d) $(-1)a = -a$.

1.2. Let F be a field, and $F' \subset F$ is a subfield. Show that F is an F'-vector space.

1.3. Which of the following sets are $\mathbb{R}$-vector spaces?

(a) Polynomials with real coefficients of degree at most n;

(b) continuous functions $f(x)$ on $\mathbb{R}$ with $f(0) = 0$;

(c) continuous functions $f(x)$ on $\mathbb{R}$ with $f(0) = 1$;

(d) unbounded functions on $\mathbb{R}$;

(e) arithmetic progressions with real entries;

(f) geometric progressions with real entries;

(g) infinite sequences of real numbers with finitely many non-zero elements;

($\star$) the set of functions $\{f(x) = a\sin(x+c) \mid a, c \in \mathbb{R}\}$.

1.4. Which of the following sets are $\mathbb{Q}$-vector spaces?

(a) Real numbers;

(b) rational numbers with denominator $\leq N$;

(c) rational numbers with denominator $\geq N$;

(d) complex numbers with rational real and imaginary parts.

1.5. Are the following sets of vectors linearly independent (over $\mathbb{R}$)?

(a) $\{(1,-1,0),\ (-1,0,1),\ (0,1,-1)\} \subset \mathbb{R}^3$;

(b) $\{(1,1,0),\ (1,0,1),\ (0,1,1)\} \subset \mathbb{R}^3$;

(c) $\{x^3+x^2,\ x^2-3x+1,\ x^3-2x^2+1\} \subset \mathbb{R}[x]$, where $\mathbb{R}[x]$ is the space of polynomials with $\mathbb{R}$-coefficients;

(d) $\{\sin x, \sin 2x, \dots, \sin nx\}$ as real functions.

1.6. For which λ

(a) linear independence of vectors $\{u, v\}$ implies linear independence of vectors $\{\lambda u, \lambda v\}$?

(b) linear independence of vectors $\{v_1, \dots, v_n\}$ implies linear independence of vectors

$$\{v_1+v_2, v_2+v_3, \dots, v_{n-1}+v_n, v_n+\lambda v_1\}?$$

1.7. ($\star$) Let V be an abelian group of all n-tuples $(z_1, \dots, z_n)$, $z_i \in \mathbb{C}$. If $\alpha \in \mathbb{C}$, define

$$\alpha \circ (z_1, \dots, z_n) = (\alpha\bar{z}_1, \dots, \alpha\bar{z}_n)$$

Is V a $\mathbb{C}$-vector space (with respect to operations $''+''$ and $''\circ''$)?