Jacobs University
School of Engineering and Science Fall Term 2011

Linear Algebra I, Homework 10

Due Date: Friday, December 2, in class.

Problems marked ($\star$) are bonus ones.

10.1. Compute exponent of the following matrices

$$(a)\ \begin{pmatrix} 3 & -1 \\ 1 & 1 \end{pmatrix} \qquad (b)\ \begin{pmatrix} 4 & -2 \\ 6 & -3 \end{pmatrix} \qquad (c)\ \begin{pmatrix} 4 & 2 & -5 \\ 6 & 4 & -9 \\ 5 & 3 & -7 \end{pmatrix}$$

10.2. Compute minimal polynomials of the following matrices

$$(a)\ \begin{pmatrix} 6 & -9 & 5 & 4 \\ 7 & -13 & 8 & 7 \\ 8 & -17 & 11 & 8 \\ 1 & -2 & 1 & 3 \end{pmatrix} \qquad (b)\ \begin{pmatrix} 0 & 1 & 0 & 0 & \dots & 0 \\ 0 & 0 & 1 & 0 & \dots & 0 \\ 0 & 0 & 0 & 1 & \dots & 0 \\ \dots & \dots & \dots & \dots & \dots & \dots \\ 0 & 0 & 0 & 0 & \dots & 1 \\ 1 & 0 & 0 & 0 & \dots & 0 \end{pmatrix}$$

10.3. Solve the following system of linear differential equations

$$\begin{cases} \mathrm{x}'(t) &= \mathrm{x(t)} & & & - & \mathrm{3z(t)} \\ \mathrm{y}'(t) &= \mathrm{x(t)} & - & \mathrm{y(t)} & - & \mathrm{6z(t)} \\ \mathrm{z}'(t) &= \mathrm{-x(t)} & + & \mathrm{2y(t)} & + & \mathrm{5z(t)} \end{cases}$$

with initial conditions

$$x(0) = 1, \qquad y(0) = 1, \qquad z(0) = 2$$

10.4. Let $f \in \mathrm{End}(V)$, $\parallel f \parallel < 1$. Define

$$\log(\mathrm{id} + f) = \sum_{n=1}^{\infty} (-1)^{n+1} \frac{f^n}{n}$$

Show that

(a) $\log(\mathrm{id} + f)$ is well defined (i.e. series converges);

(b) $e^{\log(\mathrm{id}+f)} = \mathrm{id} + f$.

10.5. Define linear Lie algebras $\mathfrak{u}(n)$ and $\mathfrak{su}(n)$ by

$$\mathfrak{u}(n) = \{A \in \mathrm{M}_n(\mathbb{C}) \mid A + \bar{A}^T = 0\}, \qquad \mathfrak{su}(n) = \mathfrak{u}(n) \cap \mathfrak{sl}(n)$$

Show that exponential map takes $\mathfrak{u}(n)$ and $\mathfrak{su}(n)$ to $U(n)$ and $SU(n)$ respectively.

10.6. ($\star$) Let $f \in \mathrm{End}(V)$, $\lambda_1, \dots, \lambda_n$ are eigenvalues of f. Show that $\parallel f \parallel \geq |\lambda_i|$ for every i.