Jacobs University
School of Engineering and Science

Fall Term 2011

Linear Algebra I, Homework 2

Due Date: Friday, September 23, in class.

Problems marked ($\star$) are bonus ones.

2.1. Let V be a vector space, $U, W \subset V$ are vector subspaces.

(a) Show that the intersection $U \cap W$ is a vector subspace of V.

(b) Denote by $U + W$ the set $\{u + w \,|\, u \in U,\ w \in W\}$. Show that $U + W$ is a vector subspace of V.

(c) Assuming that V is finite-dimensional, show that there exist bases $\{u_i\}$ of U and $\{w_j\}$ of W, such that there intersection is a basis of $U \cap W$, and union (after removing repetitions) is a basis of $U + W$.

2.2. Which of the following maps from $\mathbb{R}^3$ to $\mathbb{R}^2$ are linear maps?

(a) $f(x, y, z) = (xyz, x + y + z)$;

(b) $g(x, y, z) = (x + y, 2x - z)$.

2.3. Find the kernel and the image of the following linear maps:

(a) $\mathcal{A} : \mathbb{R}^n \to \mathbb{R}^n$, $\mathcal{A}v = 0$;

(b) $\mathcal{A} : \mathbb{C}^n \to \mathbb{C}^n$, $\mathcal{A}v = v$;

(c) $\mathcal{A} : \mathbb{R}^3 \to \mathbb{R}^3$, $\mathcal{A}(x, y, z) = (0, z, x)$;

(d) $\mathcal{A} : \mathbb{R}^4 \to \mathbb{R}^3$, $\mathcal{A}(x, y, z, t) = (x + y, y - z, z - t)$;

($\star$) $\mathcal{A} : \mathbb{R}[x] \to \mathbb{R}[x]$, $(\mathcal{A}p)(x) = p(\alpha x^2 + \beta)$, where $\alpha, \beta \in \mathbb{R}$ are fixed numbers.

2.4. Let $\mathcal{A} : V \to W$ be a surjective linear map, and let $U \subset W$ be a subspace. Show that the preimage

$$\mathcal{A}^{-1}(U) = \{v \in V \,|\, \mathcal{A}(v) \in U\}$$

is a linear subspace of V.

2.5. Let $\mathcal{A} : V \to W$ be an isomorphism (i.e., bijective linear map). Show that the inverse map $\mathcal{A}^{-1}$ is linear.

2.6. ($\star$) Find dimensions of linear spaces from Exercises 1.3 and 1.4 (or show that they do not have finite bases).