Jacobs University
School of Engineering and Science

Fall Term 2011

Linear Algebra I, Homework 3

Due Date: Friday, September 30, in class.

Problems marked ($\star$) are bonus ones.

3.1. Compute matrices of linear maps from Problem 2.3 in some bases. Which of the matrices do not depend on bases?

3.2. Let V be the space of polynomials with real coefficients of degree at most 2, and $\alpha \in \mathbb{R}$ is a fixed number, $\alpha \neq 0$. Define two maps $\mathcal{A}, \mathcal{B} : V \to V$ by

$$(\mathcal{A}p)(x) = p(\alpha x); \qquad (\mathcal{B}p)(x) = p(x + \alpha).$$

Show that $\mathcal{A}, \mathcal{B}$ are linear operators, and compute the matrices of $\mathcal{A}$ and $\mathcal{B}$ in the basis $\{1, x, x^2\}$.

3.3. Define $A = \begin{pmatrix} 1 & 0 & 1 \\ 1 & 1 & 1 \\ 0 & -2 & 1 \end{pmatrix}$, $B = \begin{pmatrix} 2 & 0 & 1 \\ -1 & 2 & 3 \end{pmatrix}$, $C = \begin{pmatrix} 2 & 1 & 1 \end{pmatrix}$. Compute the following products if possible:

(a) AB; (b) BA; (c) CA; (d) C^2; (e) A^2; (f) AC; (g) CB; (h) CAB.

3.4. Let U, V be finite-dimensional, $\mathcal{A} \in \mathrm{Hom}(U, V)$. Show that there exist bases of U and V such that the matrix A of $\mathcal{A}$ in these bases will have the following form: $a_{ii} = 1$ for $1 \le i \le \dim \operatorname{im} \mathcal{A}$, and all the other elements of A are zeros.

3.5. Compute the matrix of the operator A^m (for arbitrary $m \in \mathbb{N}$) for the following A:

(a) $\begin{pmatrix} 2 & 1 \\ 0 & 2 \end{pmatrix}$ (b) $\begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{pmatrix}$ (c) $\begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix}$ (d) $\begin{pmatrix} \lambda_1 & 0 & \dots & 0 \\ 0 & \lambda_2 & \dots & 0 \\ \dots & \dots & \dots & \dots \\ 0 & 0 & \dots & \lambda_n \end{pmatrix}$.

3.6. (a) Find all the matrices $X \in M_2$ commuting with $A = \begin{pmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{pmatrix}$.

($\star$) Find all the matrices $X \in M_n$ commuting with the matrix from Problem 2.5(d).

3.7. Find all the solutions x of the following linear systems

$$(a) \left\{ \begin{array}{rcrcrcr} x_1 & + & x_2 & + & 2x_3 & = & 1 \\ 2x_1 & - & x_2 & & & = & 3 \end{array} \right. \qquad (b) \left\{ \begin{array}{rcrcrcr} x_1 & - & x_2 & & & = & 3 \\ & + & 2x_2 & - & x_3 & = & 10 \\ 3x_1 & + & 2x_2 & + & 5x_3 & = & -2 \end{array} \right.$$

$$(c) \left\{ \begin{array}{rcrcrcr} 2x_1 & - & x_2 & + & x_3 & = & 3 \\ -x_1 & + & 2x_2 & - & x_3 & = & -5 \\ x_1 & + & x_2 & & & = & 2 \end{array} \right. \qquad (d) \left\{ \begin{array}{rcrcrcr} 2x_1 & - & x_2 & + & x_3 & = & 5 \\ -x_1 & + & 2x_2 & - & x_3 & = & -4 \\ x_1 & + & x_2 & & & = & 1 \end{array} \right.$$