Jacobs University Fall Term 2011
School of Engineering and Science

Linear Algebra I, Homework 4

Due Date: Friday, October 7, in class.

Problems marked ($\star$) are bonus ones.

4.1. Solve the following system of linear equations for every a:

$$\begin{cases} (1+a)x_1 + x_2 + x_3 = a^2+3a \\ x_1 + (1+a)x_2 + x_3 = a^3+3a^2 \\ x_1 + x_2 + (1+a)x_3 = a^4+3a^3 \end{cases}$$

4.2. Compute the rank of the following matrices

$$(a)\ \begin{pmatrix} -1 & 1 & 3 \\ 1 & 6 & 2 \\ 4 & 2 & 0 \end{pmatrix} \qquad (b)\ \begin{pmatrix} 0 & 1 & 1 \\ 3 & 2 & 3 \\ 2 & 0 & 1 \end{pmatrix} \qquad (c)\ \begin{pmatrix} 1 & -1 & 1 & 0 \\ 3 & 2 & 1 & 1 \\ 4 & 0 & 2 & -1 \end{pmatrix} \qquad (d)\ \begin{pmatrix} 2 & 0 & 3 \\ 2 & 1 & 1 \\ 3 & 0 & -1 \\ 0 & 1 & 0 \end{pmatrix}$$

4.3. Are the following matrices invertible? If yes, compute the inverses.

$$(a)\ \begin{pmatrix} 1 & 2 & 1 \\ 1 & 1 & 1 \\ 1 & 1 & 0 \end{pmatrix} \qquad (b)\ \begin{pmatrix} 1 & 1 & 1 \\ 2 & 1 & 3 \\ 4 & 3 & 5 \end{pmatrix} \qquad (c)\ \begin{pmatrix} 1 & 0 & 1 \\ 3 & 3 & 1 \\ 4 & 0 & 2 \end{pmatrix} \qquad (d)\ \begin{pmatrix} 2 & 1 & 3 & 0 \\ 0 & 1 & -2 & 1 \\ 3 & 0 & 0 & -1 \\ 0 & 1 & 1 & 0 \end{pmatrix}$$

4.4. Let $A, B \in \mathrm{M}_{m\times n}$, and assume that the homogeneous systems $Ax = 0$ and $Bx = 0$ have the same set of solutions. Show that A can be transformed into B by elementary row transformations.

4.5. Compute determinants of the following matrices using the definition of determinant only:

$$(a)\ \begin{pmatrix} 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 \end{pmatrix} \qquad (b)\ \begin{pmatrix} 0 & 0 & 1 & -1 & 0 \\ 1 & -1 & 0 & 0 & 0 \\ 0 & 1 & -1 & 0 & 0 \\ 0 & 0 & 0 & 1 & -1 \\ 1 & 1 & 0 & 0 & 0 \end{pmatrix}$$

4.6. (a) Show that if A is an invertible matrix, then A^n is also invertible for any positive integer n.

(b) Is it true that if A^n is invertible for some positive integer n then A is invertible?

(c) Find matrices $A \in \mathrm{M}_{n\times m}$ and $B \in \mathrm{M}_{m\times n}$ such that $AB = I$, but $BA \neq I$.

4.7. ($\star$) Let $\dim V = n$, and let $C \subset \mathrm{End}(V)$ be a finite set such that every $f \in \mathrm{End}(V)$ is a linear combination of products of elements of C. What is the minimal possible number of elements of C?