Jacobs University
School of Engineering and Science

Fall Term 2011

Linear Algebra I, Homework 5

Due Date: Friday, October 21, in class.

Problems marked ($\star$) are bonus ones.

5.1. Compute determinants of the following matrices:

(a) $\begin{pmatrix} 0 & 1 & 2 & 3 \\ 1 & 2 & 3 & 4 \\ 2 & 3 & 4 & 5 \\ 3 & 4 & 5 & 6 \end{pmatrix}$ (b) $\begin{pmatrix} x & y & 0 & \dots & 0 & 0 \\ 0 & x & y & \dots & 0 & 0 \\ \dots & \dots & \dots & \dots & \dots & \dots \\ 0 & 0 & 0 & \dots & x & y \\ y & 0 & 0 & \dots & 0 & x \end{pmatrix}$ (c) $\begin{pmatrix} x & y & y & \dots & y & y \\ y & x & y & \dots & y & y \\ y & y & x & \dots & y & y \\ \dots & \dots & \dots & \dots & \dots & \dots \\ y & y & y & \dots & y & x \end{pmatrix}$

5.2. Let $A \in \mathrm{M}_n(\mathbb{C})$, and let $a_{ij} = -a_{ji}$ for all i, j (i.e., A is *skew-symmetric*).

(a) Show that if n is odd then $\det A = 0$.

(b) Let n be even. Choose arbitrary complex number c, and define a matrix B by $b_{ij} = a_{ij} + c$. Show that $\det B = \det A$.

5.3. Let A be skew-symmetric with $a_{ii+1} = 1$, $a_{ij} = 0$ if $|i - j| \neq 1$. Compute $\det A$.

5.4. For $A \in \mathrm{M}_n$ define B by $b_{ij} = (-1)^{i+j} a_{ij}$. Show that $\det A = \det B$.

5.5. Let matrix A be composed of blocks:

$$A = \begin{pmatrix} B & C \\ 0 & D \end{pmatrix},$$

where B amd D are square matrices. Show that $\det A = \det B \det D$.

5.6. Let $A \in \mathrm{M}_n$ be a non-degenerate matrix with integer entries. Show that A^{-1} is also integer if and only if $\det A = \pm 1$.

5.7. Let $Ax = b$ be a system of linear equations, denote by A_i the matrix A with i-th column substituted by vector b. Show that

(a) if $\det A = 0$ but $\det A_i \neq 0$ for some i, then the system is incompatible (i.e., has no solutions);

(b) if $\det A = \det A_1 = \dots = \det A_n = 0$, then the system can be either compatible or not.

5.8. ($\star$) Let $A \in \mathrm{M}_n$, $a_{ij} = a_i b_j$ for $i \neq j$, and $a_{ii} = x_i$. Compute $\det A$.

5.9. ($\star$) Let $A \in \mathrm{M}_n$, $n \geq 3$. Show that the expansion of the determinant contains negative terms.