Jacobs University
School of Engineering and Science

Fall Term 2011

Linear Algebra I, Homework 6

Due Date: Friday, October 28, in class.

Problems marked ($\star$) are bonus ones.

6.1. Find sums and intersections of

(a) subspace of odd real functions and subspace of even real functions;

(b) subspaces V_1 and V_2 of real functions vanishing on M_1 and M_2 respectively;

(c)($\star$) subspaces P_1 and P_2 of the space $\mathbb{R}[x]$ of polynomials with real coefficients that are multiples of fixed polynomials $p_1(x)$ and $p_2(x)$ respectively.

6.2. Let $A, B, C, D \in \mathrm{M}_n$. Show that

(a) if C, D are non-degenerate then $\mathrm{rk}\,(CAD) = \mathrm{rk}\,(A)$;

(b) $\mathrm{rk}\,(AB) \leq \min(\mathrm{rk}\,(A), \mathrm{rk}\,(B))$;

(c) $\mathrm{rk}\,(A) - \mathrm{rk}\,(B) \leq \mathrm{rk}\,(A+B) \leq \mathrm{rk}\,(A) + \mathrm{rk}\,(B)$;

(d)($\star$) $\mathrm{rk}\,(BA) + \mathrm{rk}\,(AC) \leq \mathrm{rk}\,(A) + \mathrm{rk}\,(BAC)$.

6.3. Let matrix A be composed of blocks:

$$A = \begin{pmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{pmatrix},$$

where A_{11} amd A_{22} are square matrices of size m and n respectively. Let $D \in \mathrm{M}_m$, $B \in \mathrm{M}_{m\times n}$. Show that

(a) $\det \begin{pmatrix} DA_{11} & DA_{12} \\ A_{21} & A_{22} \end{pmatrix} = \det D \cdot \det A$;

(b) $\det \begin{pmatrix} A_{11} & A_{12} \\ A_{21} + BA_{11} & A_{22} + BA_{12} \end{pmatrix} = \det A$

6.4. (a) Let V be the space of real polynomials vanishing in 1. Show that $\mathbb{R}[x]/V$ is isomorphic to $\mathbb{R}$.

(b) Show that the quotient space of the space of all real sequences modulo the subspace of sequences with zero limit is isomorphic to $\mathbb{R}$.

6.5. Show that there exist canonical isomorphisms between

(a) $(V/U)/(W/U)$ and V/W if $U \subset W \subset V$;

(b) $U/(U \cap W)$ and $(U+W)/W$ if $U, W \subset V$.

6.6. ($\star$) Let A be a matrix. Show that rank of A is equal to minimal of the sizes of matrices B and C such that $A = BC$.