Jacobs University
School of Engineering and Science

Fall Term 2011

Linear Algebra I, Homework 7

Due Date: Friday, November 4, in class.

Problems marked ($\star$) are bonus ones.

7.1. Let $U, W \subset V$. Show that
(a) $(U+W)^{\perp} = U^{\perp} \cap W^{\perp}$; (b) $(U \cap W)^{\perp} = U^{\perp} + W^{\perp}$

7.2. Show that functionals $f_1, \dots, f_n$ (where $n = \dim V$ form a basis of V^* if and only if there is no nonzero vector $v \in V$ such that $f_i(v) = 0$ for all $i = 1, \dots, n$.

7.3. Let V be the space of polynomials of degree at most n.
(a) Choose $x_0, \dots, x_n \in \mathbb{F}$. Show that functionals $f_0, \dots, f_n$ defined as

$$f_i(p) = p(x_i)$$

form a basis of V^*. Find the dual basis of V.
(b) Choose $x_0 \in \mathbb{F}$. Show that functionals $f_0, \dots, f_n$ defined as

$$f_i(p) = p^{(i)}(x_0)$$

form a basis of V^*. Find the dual basis of V.

7.4. ($\star$) Let $f_1, \dots, f_m \in V^*$, $\dim V = n$. Consider a system of linear equations

$$f_1(x) = b_1, \dots, f_m(x) = b_m$$

(a) Show that if the system is compatible, then there exists a solution $x = (x_1, \dots, x_n)$, such that $x_i = \sum\limits_j c_{ij} b_j$, where c_{ij} do not depend on vector $b = (b_1, \dots, b_m)$.

(b) Let $m = n$, $f_i(x) = \sum\limits_{j=1}^{n} a_{ij} x_j$, $a_{ij} \in \mathbb{Q}$, and $f_1, \dots, f_n$ form a basis of V^*. Show that the system has a unique solution, and the numbers c_{ij} (see (a)) are rational.

7.5. Find all subspaces of $\mathbb{R}^2$ invariant with respect to operators given by following matrices:

$$(a)\ \begin{pmatrix} \cos\alpha & -\sin\alpha \\ \sin\alpha & \cos\alpha \end{pmatrix}; \qquad (b)\ \begin{pmatrix} \cosh\alpha & -\sinh\alpha \\ \sinh\alpha & \cosh\alpha \end{pmatrix}; \qquad (c)\ \begin{pmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{pmatrix}; \qquad (d)\ \begin{pmatrix} \lambda_1 & 1 \\ 0 & \lambda_2 \end{pmatrix}$$

Let $f \in \mathrm{End}(V)$. The space $E_\lambda = \{v \in V \mid f(v) = \lambda v\}$ is called *eigenspace* corresponding to eigenvalue λ.

7.6. Is it true that a subspace $V_1 \subset V$ is contained in some eigenspace if and only if every subspace $V_2 \subset V_1$ is invariant?

7.7. Compute the characteristic polynomial and find the eigenvalues (with multiplicities), eigenvectors, and bases of eigenspaces for the operators given by the following matrices.

$$(a)\ \begin{pmatrix} 1 & 1 \\ -1 & 3 \end{pmatrix} \qquad (b)\ \begin{pmatrix} 0 & 1 & -1 \\ 2 & 4 & 1 \\ 1 & 0 & 3 \end{pmatrix} \qquad (c)\ \begin{pmatrix} 3 & 3 & 0 \\ 0 & 1 & 1 \\ 0 & -1 & 1 \end{pmatrix} \qquad (d)\ \begin{pmatrix} 0 & 1 & 1 \\ -4 & -4 & 0 \\ 0 & 1 & 3 \end{pmatrix}$$