Jacobs University
School of Engineering and Science

Fall Term 2011

Linear Algebra I, Homework 8

Due Date: Friday, November 11, in class.

Problems marked $(\star)$ are bonus ones.

8.1. For $A \in \mathrm{M}_n$, the *trace* $\operatorname{tr} A$ of A is the sum of diagonal elements.

Show that

(a) $\operatorname{tr} A$ is functional on M_n;

(b) $\operatorname{tr} AB = \operatorname{tr} BA$;

(c) if $f \in \mathrm{End}(V)$, the trace $\operatorname{tr} A_f$ does not depend on basis.

(d) Express the second and the last coefficients of the characteristic polynomial $\chi_f(t)$ in terms of the elements of a matrix A_f.

8.2. Let $f \in \mathrm{End}(V)$ be an operator with simple spectrum, $\dim V = n$. Prove that

(a) every operator $g \in \mathrm{End}(V)$ commuting with f is a polynomial of f;

(b) the vector space of operators commuting with f has dimension n.

Are the statements true if f is diagonalizable, but the spectrum is not simple?

8.3. Let $f \in \mathrm{End}(V)$. Show that

(a) the kernel and image of f are f-invariant;

(b) every subspace of V containing the image of f is f-invariant;

(c) if $U \subset V$ is f-invariant, then the image and preimage of U are f-invariant;

(d) if f is nonsingular, then f-invariant spaces are f^{-1}-invariant;

(e)$(\star)$ every operator $f \in \mathrm{End}(\mathbb{C}^n)$ has an invariant subspace of dimension $n-1$.

8.4. (a) Does there exist operators $f, g \in \mathrm{End}(V)$ such that $fg - gf = \mathrm{id}$?

(b) Show that equality $fg - gf = f$ implies that f is singular.

8.5. Let $f, g \in \mathrm{End}(V)$. Show that characteristic polynomials of operators fg and gf coincide.

8.6. Let $f \in \mathrm{End}(V)$. Show that

(a) if $p(t)$ is a polynomial, and λ is an eihenvalue of f, then $p(\lambda)$ is an eigenvalue of $p(f)$;

(b) if λ^2 is an eigenvalue of f^2, then either λ or $-\lambda$ is an eigenvalue of f.

8.7. $(\star)$ Show that every polynomial with leading coefficient $(-1)^n$ is a characteristic polynomial of some matrix $A \in \mathrm{M}_n$.

8.8. Find all the eigenspaces and root subspaces of the operator given by matrix

$$(a)\ \begin{pmatrix} 4 & -5 & 2 \\ 5 & -7 & 3 \\ 6 & -9 & 4 \end{pmatrix} \qquad (b)\ \begin{pmatrix} 1 & -3 & 4 \\ 4 & -7 & 8 \\ 6 & -7 & 7 \end{pmatrix} \qquad (c)\ \begin{pmatrix} 2 & 6 & -15 \\ 1 & 1 & -5 \\ 1 & 2 & -6 \end{pmatrix} \qquad (d)\ \begin{pmatrix} 0 & -2 & 3 & 2 \\ 1 & 1 & -1 & -1 \\ 0 & 0 & 2 & 0 \\ 1 & -1 & 0 & 1 \end{pmatrix}$$