

Complex numbers

Practice list

1. Compute
 - a) $(-i)^2$; b) i^{10} ; c) $(1+i)^{10}$; d) $(1-i)^{10}$; e) $(1+i)^{101}$.
2. Find $z, w \in \mathbb{C}$, such that $z + w = zw = 2$.
3. Find $1/(a + bi)$.
4. Compute the sum $1 + i + i^2 + i^3 + \dots + i^{100}$.
5. Solve the following equations:
 - a) $z^2 = 2$; b) $z^2 = -2$; c) $z^2 = 2i$; d) $z^2 = 1 + i$;
 - e) $z^2 + 2z + 2 = 0$; f) $z^2 + (2i - 7)z + 13 - i = 0$.
6. Compute real and imaginary parts of $(a + bi)(c + di)$.
7. Prove that $z + \bar{z} \in \mathbb{R}$.
8. Prove that
 - a) $\overline{z + w} = \bar{z} + \bar{w}$; b) $\overline{z\bar{w}} = \bar{z} w$; c) $\overline{z/w} = \bar{z}/\bar{w}$.
9. Compute
 - a) $(1+i)/(1-i)$; b) $(8+i)/(1+2i)$; c) $(5+i)(7-6i)/(3+i)$;
 - d) $(1+i)^5/(1-i)^3$; e) $(1+3i)(8-i)/(2+i)^2$.
10. Show that if $zw \in \mathbb{R}$ and $z+w \in \mathbb{R}$ then either both z and w are real, or $z = \bar{w}$.
11. Show that $|z|^2 = z\bar{z}$.
12. Find the distance from z to w .
13. Show that $|zw| = |z||w|$.
14. Find the real and imaginary part of a complex number with absolute value r and argument φ .
15. Find the locus
 - a) $z + \bar{z} = 1$; b) $z\bar{z} = 1$; c) $|z| = 1$; d) $|z - 1| = 1$;
 - e) $|z| = |z+1|$; f) $|2z|^2 = z\bar{z}$; g) $\Re z/(z-1) = 0$; h) $|z-a| = |z-b|$.
16. Prove that the points $0, 1/z$ and \bar{z} are collinear.
17. Show that the locus $z + \bar{z} = z\bar{z}$ is a circle. Find its center and radius.
18. Find all the $z \in \mathbb{C}$, such that $|z - 3| \leq 2$ and $|z + 4i| \leq 3$.
19. Show that the points $z, w, 1/\bar{z}, 1/\bar{w}$ belong to one circle.
20. Draw the locus $z\bar{z} + az + \bar{a}\bar{z} + c = 0$, where $a \in \mathbb{C}, c \in \mathbb{R}$. How does the answer depend on a and c ?
21. Solve the following equations:
 - a) $|z| - z = 1 + 2i$; b) $|z| + z = 2 + i$.

22. Show that
 $(\cos \alpha + i \sin \alpha)^k = \cos k\alpha + i \sin k\alpha.$
23. Compute $(1 + \cos \alpha + i \sin \alpha)^k.$
24. Show that if $z + 1/z = 2 \cos \alpha$, then $z^m + 1/z^m = 2 \cos m\alpha.$
25. Write down the following complex numbers in the form $a + bi$:
 a) $\sqrt{8\sqrt{3}i - 8};$ b) $\sqrt[3]{(27 - 54i)/(2 + i)};$ c) $\sqrt[3]{1 + i}.$
26. Find an integer n such that $(\frac{2+i}{2-i})^n = 1.$
27. Compute
 a) $\cos x + \cos 2x + \cdots + \cos nx;$
 b) $\sin x + \sin 2x + \cdots + \sin nx;$
 c) $\sin^2 x + \sin^2 3x + \cdots + \sin^2(2n - 1)x;$
 d) $\cos x + 2 \cos 2x + 3 \cos 3x + \cdots + n \cos nx.$
28. Solve the following equations:
 a) $x^4 + 6x^3 + 9x^2 + 100 = 0;$ b) $x^4 + 2x^2 - 24x + 72 = 0.$
29. Compute
 a) $1 + 2\varepsilon + 3\varepsilon^2 + \cdots + n\varepsilon^{n-1},$ where $\varepsilon^n = 1;$
 b) $\varepsilon_0^k + \varepsilon_1^k + \cdots + \varepsilon_{n-1}^k$ ($\varepsilon_0, \dots, \varepsilon_{n-1}$ — n -roots of unity).