Durham University
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Michaelmas 2013

Differential Geometry III, Homework 1 (Week 1)

Due date for starred problems: **Thursday, October 24**.

Plane curves - 1

1.1. Sketch the trace of the smooth curve given by $\boldsymbol{\alpha}(u) = (u^5, u^2 - 1)$, and mark the singular points.

1.2. Let $\boldsymbol{\alpha} : I \to \mathbb{R}^2$ be a smooth curve, and let $[a, b] \subset I$ be a closed interval. For every partition $a = u_0 < u_1 < \cdots < u_n = b$ consider the sum

$$\ell_{\boldsymbol{\alpha},P} := \sum_{i=1}^{n} \|\boldsymbol{\alpha}(u_i) - \boldsymbol{\alpha}(u_{i-1})\|$$

where P stands for the given partition. Give a geometric interpretation of $\ell_{\boldsymbol{\alpha},P}$. What length does $\ell_{\boldsymbol{\alpha},P}$ measure? Now assume that the partition becomes *finer*, i.e., $\|P\| := \max_{i=1,\dots,n} |u_i - u_{i-1}|$ becomes smaller. What is the limit of $\ell_{\boldsymbol{\alpha},P}$ as $\|P\| \to 0$?

1.3. ($\star$) An *epicycloid* $\boldsymbol{\alpha}$ is obtained as the locus of a point on the circumference of a circle of radius r which rolls without slipping on a circle of the same radius.

(a) Sketch $\boldsymbol{\alpha}$.

(b) Show that the epicycloid may be parametrized by

$$\boldsymbol{\alpha}(u) = (2r \sin u - r \sin 2u,\ 2r \cos u - r \cos 2u), \qquad u \in \mathbb{R}.$$

Find the length of $\boldsymbol{\alpha}$ between the singular points at $u = 0$ and $u = 2\pi$.

1.4. ($\star$) (a) Let $\boldsymbol{\alpha}(u)$ and $\boldsymbol{\beta}(u)$ be two smooth plane curves. Show that

$$\frac{d}{du}(\boldsymbol{\alpha}(u) \cdot \boldsymbol{\beta}(u)) = \boldsymbol{\alpha}'(u) \cdot \boldsymbol{\beta}(u) + \boldsymbol{\alpha}(u) \cdot \boldsymbol{\beta}'(u),$$

where $\boldsymbol{\alpha}(u) \cdot \boldsymbol{\beta}(u)$ denotes a Euclidean dot product of vectors $\boldsymbol{\alpha}(u)$ and $\boldsymbol{\beta}(u)$.

Hint: write $\boldsymbol{\alpha}(u) = (\alpha_1(u), \alpha_2(u))$, $\boldsymbol{\beta}(u) = (\beta_1(u), \beta_2(u))$ and compute everything in coordinates.

(b) Let $\boldsymbol{\alpha}(u) : I \to \mathbb{R}^2$ be a smooth curve which does not pass through the origin. Suppose there exists $u_0 \in I$ such that the point $\boldsymbol{\alpha}(u_0)$ is the closest to the origin amongst all the points of the trace of $\boldsymbol{\alpha}$. Show that $\boldsymbol{\alpha}(u_0)$ is orthogonal to $\boldsymbol{\alpha}'(u_0)$.

1.5. The second derivative $\boldsymbol{\alpha}''(u)$ of a smooth plane curve $\boldsymbol{\alpha}(u)$ is identically zero. What can be said about $\boldsymbol{\alpha}$?

1.6. Let $\boldsymbol{\alpha} : (0, \pi) \to \mathbb{R}^2$ be a curve defined by

$$\boldsymbol{\alpha}(u) = (\sin u, \cos u + \log \tan \frac{u}{2})$$

The trace of $\boldsymbol{\alpha}$ is called a *tractrix*.

(a) Sketch $\boldsymbol{\alpha}$.

(b) Show that a tangent vector at $\boldsymbol{\alpha}(u_0)$ can be written as

$$\boldsymbol{\alpha}'(u_0) = (\cos u_0, -\sin u_0 + \frac{1}{\sin u_0})$$

Show that $\alpha(u)$ is smooth, and it is regular everywhere except $u = \pi/2$.

(c) Write down the equation of a tangent line l_{u_0} to the trace of $\boldsymbol{\alpha}$ at $\boldsymbol{\alpha}(u_0)$.

(d) Show that the distance between $\boldsymbol{\alpha}(u_0)$ and the intersection of l_{u_0} with y-axis is constantly equal to 1.