Durham University
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Differential Geometry III, Homework 10 (Week 10)

Coordinate curves, angles and area

10.1. Let $\boldsymbol{x} : U \to S$ be a local parametrization of a regular surface S, and denote by E, F, G the coefficients of the first fundamental form in this parametrization. Show that the tangent vector $a\,\partial_u\boldsymbol{x} + b\,\partial_v\boldsymbol{x}$ bisects the angle between the coordinate curves if and only if

$$\sqrt{G}(aE + bF) = \sqrt{E}(aF + bG).$$

Further, if

$$\boldsymbol{x}(u, v) = (u, v, u^2 - v^2),$$

find a vector tangential to S which bisects the angle between the coordinate curves at the point $(1, 1, 0) \in S$.

10.2. Find two families of curves on the helicoid parametrized by

$$\boldsymbol{x}(u, v) = (v \cos u, v \sin u, u)$$

which, at each point, bisect the angles between the coordinate curves.

(Show that they are given by $u \pm \sinh^{-1} v = c$, where c is a constant on each curve in the family.)

10.3. Let S be the helicoid defined in Exercise 10.2. Find the area of the part of S defined by

$$1 < v < 3, \qquad 0 < u < 6\pi$$

10.4. The coordinate curves of a parametrization $\boldsymbol{x}(u, v)$ constitute a *Chebyshev net* if the lengths of the opposite sides of any quadrilateral formed by them are equal.

(a) Show that a necessary and sufficient condition for this is

$$\frac{\partial E}{\partial v} = \frac{\partial G}{\partial u} = 0$$

(b) Show that if coordinate curves constitute a Chebyshev net, then it is possible to reparametrize the coordinate neighborhood in such a way that the new coefficients of the first fundamental form are

$$E = 1, \qquad F = \cos\vartheta, \qquad G = 1,$$

where ϑ is the angle between coordinate curves.

10.5. Show that a surface of revolution can always be parametrized so that

$$E = E(v), \qquad F = 0, \qquad G = 1$$

10.6. Let S be the surface $\{(x, y, z) \in \mathbb{R}^3 \mid z = x^2 - y^2\}$ and let $\mathcal{F}$ be the family of curves on S obtained as the intersection of S with the planes $z =$ const. Find the family of curves on S which meet $\mathcal{F}$ orthogonally and show that they are the intersections of S with the family of hyperbolic cylinders $xy =$ const.

10.7. Using the notation of Exercise 10.2, show that the family of curves orthogonal to the family

$$v \cos u = \text{const}$$

is the family defined by $(1 + v^2)\sin^2 u = \text{const}$.