Durham University
Pavel Tumarkin Michaelmas 2013

Differential Geometry III, Homework 4 (Week 4)

Due date for starred problems: **Thursday, November 7**.

Space curves - 1

4.1. Let $\boldsymbol{\alpha} : I \to \mathbb{R}^3$ be a regular space curve. Show that the curvature κ and the torsion τ of $\boldsymbol{\alpha}$ are given by

$$\kappa = \frac{\|\boldsymbol{\alpha}' \times \boldsymbol{\alpha}''\|}{\|\boldsymbol{\alpha}'\|^3}, \qquad \tau = -\frac{(\boldsymbol{\alpha}' \times \boldsymbol{\alpha}'') \cdot \boldsymbol{\alpha}'''}{\|\boldsymbol{\alpha}' \times \boldsymbol{\alpha}''\|^2}.$$

Hint: Denote the unit tangent, unit principal normal and binormal vector of $\boldsymbol{\alpha}$ by $\boldsymbol{t}$, $\boldsymbol{n}$ and $\boldsymbol{b}$. Introduce the corresponding curve $\widetilde{\boldsymbol{\alpha}}$ parametrized by arc length $s = \ell(u)$ (i.e., $\widetilde{\boldsymbol{\alpha}}(\ell(u)) = \boldsymbol{\alpha}(u)$), and denote the corresponding orthonormal frame by $\widetilde{\boldsymbol{t}}$, $\widetilde{\boldsymbol{n}}$ and $\widetilde{\boldsymbol{b}}$. You might find it useful to write $(\cdot)'$ for the derivative with respect to u and $\frac{\mathrm{d}}{\mathrm{d}s}$ for the derivative with respect to s, and to use the short hand notation

$$\frac{\mathrm{d}}{\mathrm{d}s} = \frac{1}{\|\boldsymbol{\alpha}'\|}(\cdot)'.$$

You may also need the fact that $(\boldsymbol{a} \times \boldsymbol{b}) \cdot \boldsymbol{c} = (\boldsymbol{b} \times \boldsymbol{c}) \cdot \boldsymbol{a} = (\boldsymbol{c} \times \boldsymbol{a}) \cdot \boldsymbol{b}$ *(triple product)*.

4.2. ($\star$) Find the curvature and torsion of the curve

$$\boldsymbol{\alpha}(u) = (au, bu^2, cu^3).$$

4.3. ($\star$) Assume that $\boldsymbol{\alpha} : I \to \mathbb{R}^3$ is a regular space curve parametrized by arc length.

(a) Determine all regular curves with vanishing curvature κ.

Hint: use Exercise 4.1

(b) Show that if the torsion τ of $\boldsymbol{\alpha}$ vanishes, then the trace of $\boldsymbol{\alpha}$ lies in a plane.

Hint: do NOT use Exercise 4.1

4.4. Assume that $\boldsymbol{\alpha}(s) = (x(s), y(s), 0)$, i.e., the trace of $\boldsymbol{\alpha}$ lies in the plane $z = 0$. Calculate the curvature κ of $\boldsymbol{\alpha}$ and its torsion τ. What is the relation of the curvature κ of the space curve $\boldsymbol{\alpha}$ and the (signed) curvature $\overline{\kappa}$ of the plane curve $\overline{\boldsymbol{\alpha}} : I \to \mathbb{R}^2$ defined by $\overline{\boldsymbol{\alpha}}(s) = (x(s), y(s))$ (i.e., the projection of the space curve $\boldsymbol{\alpha}$ to the plane $z = 0$)?

4.5. Consider the regular curve given by

$$\boldsymbol{\alpha}(s) = \left(a\cos\frac{s}{c}, a\sin\frac{s}{c}, b\frac{s}{c}\right), \qquad s \in \mathbb{R},$$

where $a, b, c > 0$ and $c^2 = a^2 + b^2$. The curve $\boldsymbol{\alpha}$ is called a *helix*.

(a) Show that the trace of $\boldsymbol{\alpha}$ lies on the cylinder $x^2 + y^2 = a^2$.

(b) Show that $\boldsymbol{\alpha}$ is parametrized by arc length.

(c) Determine the curvature and torsion of $\boldsymbol{\alpha}$ (and notice that they are both constant).

(d) Determine the equation of the plane through $\boldsymbol{n}(s)$ and $\boldsymbol{t}(s)$ at each point of $\boldsymbol{\alpha}$ (this plane is called the *osculating plane*).

(e) Show that the line through $\boldsymbol{\alpha}(s)$ in direction $\boldsymbol{n}(s)$ meets the axis of the cylinder orthogonally.

(f) Show that the tangent lines to $\boldsymbol{\alpha}$ make a constant angle with the axis of the cylinder.

Remark: In fact, a helix can be characterized by (a) and (f). If we drop (a), then we obtain a *generalized helix* (see next homework). Another way how to characterize a helix is by (c), i.e., the fact that the curvature and torsion are constant. *Why?*