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Michaelmas 2013

Differential Geometry III, Homework 6 (Week 6)

Due date for starred problems: **Thursday, November 21**.

Surfaces - 1

6.1. Let $U \subset \mathbb{R}^2$ be an open set. Show that the set

$$\{(x, y, z) \in \mathbb{R}^3 \,|\, z = 0 \text{ and } (x, y) \in U\}$$

is a regular surface.

6.2. ($\star$) **Stereographic projection**
Let $S^2(1) = \{(x, y, z) \in \mathbb{R}^3 \,|\, x^2 + y^2 + z^2 = 1\}$ be a 2-dimensional unit sphere. For $(u, v) \in \mathbb{R}^2$, let $\boldsymbol{x}(u, v)$ be the point of intersection of the line in $\mathbb{R}^3$ through $(u, v, 0)$ and $(0, 0, 1)$ with $S^2(1)$ (different from $(0, 0, 1)$).

(a) Find an explicit formula for $\boldsymbol{x}(u, v)$.

(b) Let P be the plane given by $\{z = 1\}$, and for $(x, y, z) \in \mathbb{R}^3 \setminus P$, let $\boldsymbol{F}(x, y, z) \in \mathbb{R}^2$ be such that $(\boldsymbol{F}(x, y, z), 0) \in \mathbb{R}^3$ is the intersection with the (x, y)-plane of the line through $(0, 0, 1)$ and (x, y, z). Show that

$$\boldsymbol{F}(x, y, z) = \frac{1}{1 - z}(x, y).$$

(c) Show that $\boldsymbol{F} \circ \boldsymbol{x} = \text{id} : \mathbb{R}^2 \to \mathbb{R}^2$ and deduce that $\boldsymbol{x}$ is a local parametrization of $S^2(1) \setminus \{(0, 0, 1)\}$.

6.3. Show that each of the following is a surface:

(a) a cylinder $\{(x, y, z) \in \mathbb{R}^3 \,|\, x^2 + y^2 = 1\}$;

(b) a two-sheet hyperboloid given by $\{(x, y, z) \in \mathbb{R}^3 \,|\, x^2 + y^2 - z^2 = -1\}$.

In each case find a covering of the surface by coordinate neighborhoods and give a sketch of the surface indicating the coordinate neighbourhoods you have used.

6.4. For $a, b > 0$, let

$$S := \left\{ (x, y, z) \in \mathbb{R}^3 \;\middle|\; z = \frac{x^2}{a^2} - \frac{y^2}{b^2} \right\}.$$

Show that S is a surface and show that at each point $p \in S$ there are two straight lines passing through p and lying in S (i. e. S is a *doubly ruled* surface).

6.5. Let S be the surface in $\mathbb{R}^3$ defined by $z = x^2 - y^2$. Show that

$$\boldsymbol{x}(u, v) = (u + \cosh v, u + \sinh v, 1 + 2u(\cosh v - \sinh v)), \qquad u, v \in \mathbb{R},$$

is a local parametrization of S.

6.6. Show that

(a) the cone $\{x^2 + y^2 - z^2 = 0\}$ is not a regular surface;

(b) the one-sheet cone $\{x^2 + y^2 - z^2 = 0,\; z \geq 0\}$ is not a regular surface.

Hint: in (b) you need to prove that for *every* parametrization of the neighborhood of the origin the regularity condition fails.