Durham University
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Differential Geometry III, Homework 7 (Week 7)

Due date for starred problems: **Thursday, December 5**.

Surfaces - 2

7.1. ($\star$) (a) Parametrize the hyperbolic paraboloid S from Exercise 6.4 as a ruled surface (i.e., find a curve $\boldsymbol{\alpha}(v) \subset S$ and a curve $\boldsymbol{w}(v)$ such that $\boldsymbol{x}(u,v) = \boldsymbol{\alpha}(v) + u\boldsymbol{w}(v)$ will be a parametrization of S).

(b) Now let S be an arbitrary ruled surface, and let $\boldsymbol{x} : J \times I \to \mathbb{R}^3$, $\boldsymbol{x}(u,v) = \boldsymbol{\alpha}(v) + u\boldsymbol{w}(v)$ be a parametrization of S such that $|\boldsymbol{w}(v)| = 1$ for all $v \in I$, where $\boldsymbol{\alpha} : I \to \mathbb{R}^3$ is a regular space curve and I, J are intervals in $\mathbb{R}$. A curve $\boldsymbol{\beta} : I \to \mathbb{R}^3$ lying in S is called a *curve of striction* if $\boldsymbol{\beta}'(v) \cdot \boldsymbol{w}'(v) = 0$ for all $v \in I$. Find the curve of striction of the ruled surface in (a) with $a = b = 1$ (using either one of the rulings).

Hint: You may assume $\boldsymbol{\beta}(v) = \boldsymbol{\alpha}(v) + u(v)\boldsymbol{w}(v)$.

7.2. (a) Show that the set S of $(x, y, z) \in \mathbb{R}^3$ fulfilling the equation $xz + y^2 = 1$ is a surface.

(b) Let $\boldsymbol{\alpha}, \boldsymbol{w} : \mathbb{R} \to \mathbb{R}^3$ be given by

$$\boldsymbol{\alpha}(v) = (\cos v, \sin v, \cos v) \quad \text{and} \quad \boldsymbol{w}(v) = (1 + \sin v, -\cos v, -1 + \sin v).$$

Show that for all $v \in \mathbb{R}$ there are two straight lines through $\boldsymbol{\alpha}(v)$, one of which is in direction $\boldsymbol{w}(v)$, both of which lie on S. If $\boldsymbol{x}(u,v) = \boldsymbol{\alpha}(v) + u\boldsymbol{w}(v)$, $u \in \mathbb{R}$, $0 < v < 2\pi$, show that $\boldsymbol{x}$ is a local parametrization of S.

7.3. Determine all surfaces of revolution which are also ruled surfaces.

7.4. ($\star$) Let $f : \mathbb{R}^3 \to \mathbb{R}$ be given by $f(x, y, z) = (x + y + z - 1)^2$.

(a) Find the points at which $\operatorname{grad} f = 0$.

(b) For which values of c the level set $S := \{p = (x, y, z) \in \mathbb{R}^3 \mid f(p) = c\}$ is a surface?

(c) What is the level set $f(p) = c$?

(d) Repeat (a) and (b) using the function $f(x, y, z) = xyz^2$.

7.5. Möbius band
Let S be the image of the function $f : \mathbb{R} \times (-\varepsilon, \varepsilon) \to \mathbb{R}^3, (\varepsilon > 0)$, defined by

$$f(u,v) = \left(\left(2 - v\sin\frac{u}{2}\right)\sin u,\ \left(2 - v\sin\frac{u}{2}\right)\cos u,\ v\cos\frac{u}{2}\right).$$

Show that, for ε sufficiently small, S is a surface in $\mathbb{R}^3$ which may be covered by two coordinate neighborhoods. Give a sketch of the surface indicating the curves $u = \text{const}$ and $v = \text{const}$ (such curves are called *coordinate curves*).

7.6. Real projective plane (bonus problem)
Let $f : \mathbb{R}^3 \to \mathbb{R}^5$ be defined by

$$f(x, y, z) = \Big(yz, zx, xy, \frac{1}{2}(x^2 - y^2), \frac{1}{2\sqrt{3}}(x^2 + y^2 - 2z^2)\Big).$$

Show that:

(a) $f(x, y, z) = f(x', y', z')$ if and only if $(x, y, z) = \pm(x', y', z')$;

(b) the image $S = f(S^2(1))$ of the unit sphere $S^2(1)$ in $\mathbb{R}^3$ is a surface in $\mathbb{R}^5$.

The surface S is often written as $\mathbb{R}P^2$ and is called the *real projective plane*. Note that it can be identified with the set of lines through the origin in $\mathbb{R}^3$.