Durham University
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Michaelmas 2013

Differential Geometry III, Homework 8 (Week 8)

Due date for starred problems: **Thursday, December 5**.

Tangent plane

8.1. (a) Let $\boldsymbol{x} : U \to S$ be a local parametrization of a surface S in some neiborhood of a point $\boldsymbol{p} = (x_0, y_0, z_0) \in S$. Show that the tangent plane to S at $\boldsymbol{p}$ has equation

$$\left(\frac{\partial \boldsymbol{x}}{\partial u}(\boldsymbol{p}) \times \frac{\partial \boldsymbol{x}}{\partial v}(\boldsymbol{p})\right) \cdot (x - x_0, y - y_0, z - z_0) = 0$$

(b) Let $f : \mathbb{R}^3 \to \mathbb{R}$ be a smooth function, and let $c \in f(\mathbb{R}^3)$ be a regular value of f. Show that the tangent plane of the regular surface

$$S = \{(x, y, z) \,|\, f(x, y, z) = c\}$$

at the point $\boldsymbol{p} = (x_0, y_0, z_0) \in S$ has equation

$$\frac{\partial f}{\partial x}(\boldsymbol{p})(x - x_0) + \frac{\partial f}{\partial y}(\boldsymbol{p})(y - y_0) + \frac{\partial f}{\partial z}(\boldsymbol{p})(z - z_0) = 0$$

8.2. Show that the tangent plane of one-sheeted hyperboloid $x^2 + y^2 - z^2 = 1$ at point $(x, y, 0)$ is parallel to the z-axis.

8.3. ($\star$) Let $f : \mathbb{R} \to \mathbb{R}$ be a smooth function. Define a surface S as

$$S = \{(x, y, z) \,|\, xf(y/x) - z = 0,\ x \neq 0\}$$

Show that all tangent planes of S pass through the origin $(0, 0, 0)$.

8.4. Let $U \subset \mathbb{R}^2$ be open, and let S_1 and S_2 be two regular surfaces with parametrizations $\boldsymbol{x} : U \to S_1$ and $\boldsymbol{y} : U \to S_2$. Define a map $\boldsymbol{\varphi} = \boldsymbol{y} \circ \boldsymbol{x}^{-1} : S_1 \to S_2$. Let $\boldsymbol{p} \in S_1$, $\boldsymbol{w} \in T_{\boldsymbol{p}}S_1$, and let $\boldsymbol{\alpha} : (-\varepsilon, \varepsilon) \to S_1$ be an arbitrary regular curve in S_1 such that $\boldsymbol{p} = \boldsymbol{\alpha}(0)$ and $\boldsymbol{\alpha}'(0) = \boldsymbol{w}$. Define $\boldsymbol{\beta} : (-\varepsilon, \varepsilon) \to S_2$ as $\boldsymbol{\beta} = \boldsymbol{\varphi} \circ \boldsymbol{\alpha}$.

(a) Show that $\boldsymbol{\beta}'(0)$ does not depend on the choice of $\boldsymbol{\alpha}$.

(b) Show that the map $\mathrm{d}_{\boldsymbol{p}}\boldsymbol{\varphi} : T_{\boldsymbol{p}}S_1 \to T_{\boldsymbol{\varphi}(\boldsymbol{p})}S_2$ defined by $\mathrm{d}_{\boldsymbol{p}}\boldsymbol{\varphi}(\boldsymbol{w}) = \boldsymbol{\beta}'(0)$ is linear.

8.5. Let $\boldsymbol{\alpha} : I \to \mathbb{R}^3$ be a regular curve with nonzero curvature parametrized by arc length. Recall that a *canal surface* (or *tubular surface*) S is a surface parametrized by

$$\boldsymbol{x}(u, v) = \boldsymbol{\alpha}(u) + r(\boldsymbol{n}(u)\cos v + \boldsymbol{b}(u)\sin v),$$

where $\boldsymbol{n}$ and $\boldsymbol{b}$ are unit normal and binormal vectors, and $r > 0$ is a sufficiently small constant. Find the equation of the tangent plane to S at $\boldsymbol{x}(u, v)$. In particular, show that the tangent plane at $\boldsymbol{x}(u, v)$ is parallel to $\boldsymbol{\alpha}'(u)$.