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Differential Geometry III, Homework 9 (Week 9)

First fundamental form

9.1. Find the coefficients of the first fundamental forms of:

(a) the *catenoid* parametrized by

$$\boldsymbol{x}(u,v) = (\cosh v \cos u, \cosh v \sin u, v), \qquad (u,v) \in U := (0, 2\pi) \times \mathbb{R};$$

(b) the *helicoid* parametrized by

$$\widetilde{\boldsymbol{x}}(u,v) = (-\sinh v \sin u, \sinh v \cos u, -u), \qquad (u,v) \in U;$$

(c) the surface S_ϑ (for some $\vartheta \in \mathbb{R}$) parametrized by

$$\boldsymbol{y}_\vartheta(u,v) = (\cos\vartheta)\boldsymbol{x}(u,v) + (\sin\vartheta)\widetilde{\boldsymbol{x}}(u,v), \qquad (u,v) \in U.$$

9.2. Find the coefficients of the first fundamental form of

(a) $S^2(1)$ with respect to the local parametrization $\boldsymbol{x}$ defined in Exercise 6.2;

(b) the surface

$$S = \{(x,y,z) \in \mathbb{R}^3 \mid x \sin z - y \cos z = 0\}$$

parametrized by

$$\boldsymbol{x}(u,v) = (\sinh v \cos u, \sinh v \sin u, u)$$

9.3. Let $U = \mathbb{R} \times (0, \infty)$, and let $\boldsymbol{x} : U \to \mathbb{R}^n$ be a parametrization of a surface $\mathbb{H}$ in $\mathbb{R}^2$ with corresponding coefficients of the first fundamental form $E(u,v) = G(u,v) = 1/v^2$ and $F(u,v) = 0$ for all $(u,v) \in U$. Then $\mathbb{H}$ is called the *hyperbolic plane*. For $r > 0$ denote by $\boldsymbol{\alpha} : (0, \pi) \to \mathbb{H}$ the curve given by

$$\boldsymbol{\alpha}(t) = \boldsymbol{x}(r\cos t, r \sin t).$$

Show that the length of $\boldsymbol{\alpha}$ in $\mathbb{H}$ from $\boldsymbol{\alpha}(\pi/6)$ to $\boldsymbol{\alpha}(5\pi/6)$ is equal to

$$\int_{\pi/6}^{5\pi/6} \frac{1}{\sin t}\, \mathrm{d}t.$$

(In fact, $\boldsymbol{\alpha}$ is the curve of shortest length between its endpoints.) Now take $r = \sqrt{2}$ and find the angle of intersection of $\boldsymbol{\alpha}$ with the curve $\boldsymbol{\beta}(s) = \boldsymbol{x}(1, s)$ at their point of intersection.

9.4. Let S be a surface parametrized by

$$\boldsymbol{x}(u,v) = (u \cos v, u \sin v, \log \cos v + u), \qquad (u,v) \in U := \mathbb{R} \times \left(-\frac{\pi}{2}, \frac{\pi}{2}\right).$$

For $c \in (-\pi/2, \pi/2)$, let $\boldsymbol{\alpha}_c$ be the curve given by $\boldsymbol{\alpha}_c(u) = \boldsymbol{x}(u, c)$. Show that the length of $\boldsymbol{\alpha}_c$ from $u = u_0$ to $u = u_1$ does not depend on c.