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Riemannian Geometry IV, Solutions 2 (Week 2)

2.1. (a) Let U be an open subset of $\mathbb{R}^n$. Show that U is a smooth manifold.

(b) Show that the general linear group $GL_n(\mathbb{R}) = \{A \in M_n(\mathbb{R}) \,|\, \det A \neq 0\}$ is a smooth manifold.

Solution:

(a) The atlas consists of one chart (U, U, id).

(b) Due to (a), it is enough to show that $GL_n(\mathbb{R})$ is open in $M_n(\mathbb{R})$.

One way to do this is to show that if $\det A \neq 0$ then there exists $\varepsilon > 0$ such that for any $B \in M_n(\mathbb{R})$ satisfying $|b_{ij} - a_{ij}| < \varepsilon$ for all $i, j \leq n$ the determinant of B is not zero. Indeed, a very rough estimate gives

$$|\det B - \det A| \leq \varepsilon(n \max_{i,j\leq n}(|a_{ij}|^{n-1}) + O(\varepsilon^2)$$

which is clearly less than $|\det A|$ for ε small enough.

Alternatively, one can note that the function $\det : M_2(\mathbb{R}) \to \mathbb{R}$ is continuos (since it is a polynomial), and $GL_n(\mathbb{R})$ is the preimage of an open set $\mathbb{R} \setminus \{0\}$, so it is open.

2.2. ($\star$)

(a) Show that the set of $n \times n$ matices real matrices with positive determinant is an open subset of $M_n(\mathbb{R})$.

(b) Show that the special orthogonal group $SO_2(\mathbb{R}) = \{A \in M_2(\mathbb{R}) \,|\, A^t A = I,\ \det A = 1\}$ is a smooth 1-manifold.

Solution:

(a) See the proof of Exercise 2.1(b).

(b) If $A = \begin{pmatrix} x_1 & x_2 \\ x_3 & x_4 \end{pmatrix}$, then $A^t A = \begin{pmatrix} x_1^2 + x_3^2 & x_1x_2 + x_3x_4 \\ x_1x_2 + x_3x_4 & x_2^2 + x_4^2 \end{pmatrix}$. Define a map $f : M_2(\mathbb{R}) \to \mathbb{R}^3$,

$$f(A) = \begin{pmatrix} x_1^2 + x_3^2 \\ x_1x_2 + x_3x_4 \\ x_2^2 + x_4^2 \end{pmatrix}$$

Then the preimage of a vector $(1, 0, 1)$ will consist of all orthogonal (2×2)–matrices (the group of all orthogonal $(n \times n)$–matrices is denoted by $O_n(\mathbb{R})$). Let us show that $(1, 0, 1)$ is a regular value of f.

The differential of f at A is

$$Df(A) = \begin{pmatrix} 2x_1 & 0 & 2x_3 & 0 \\ x_2 & x_1 & x_4 & x_3 \\ 0 & 2x_2 & 0 & 2x_4 \end{pmatrix}$$

The first three columns form a submatrix of A with determinant $-8x_2 \det A$, the first two columns together with the forth one form a submatrix of A with determinant $8x_4 \det A$. Thus, if the rank of A is less than 3, then either $\det A = 0$, of $x_2 = x_4 = 0$. Since neither of this may happen for an orthogonal matrix, we see that the rank of $Df(A)$ is 3 for $A \in O_2(\mathbb{R})$, and thus $(1, 0, 1)$ is a regular value of f, which implies that $O_2(\mathbb{R})$ is a smooth manifold.

Now, $SO_2(\mathbb{R})$ is the intersection of $O_2(\mathbb{R})$ with the set of matrices with positive determinant, which is open due to (a). Applying Exercise 2.1(a), we see that an intersection of a smooth manifold with an open set is again a smooth manifold.

Alternatively, one could observe that any matrix from $SO_2(\mathbb{R})$ is of the form

$$\begin{pmatrix} \cos\vartheta & -\sin\vartheta \\ \sin\vartheta & \cos\vartheta \end{pmatrix}$$

for $\vartheta \in [0, 2\pi)$, and construct a smooth atlas explicitely (e.g., the same as was constructed for the unit circle on the first lecture).

2.3. Show that the special orthogonal group $SO_n(\mathbb{R}) = \{A \in M_n(\mathbb{R}) \,|\, A^t A = I, \ \det A = 1\}$ is a smooth manifold.

Solution:

The proof is very similar to the solution of the previous problem. If we denote the columns of a matrix $A \in M_n(\mathbb{R})$ by $a_1, \dots, a_n$, then the entries of the matrix $B = A^t A$ are $b_{ij} = \langle a_i, a_j \rangle$, where $\langle \cdot, \cdot \rangle$ is the standard dot-product. Define $f : M_n(\mathbb{R}) \to \mathbb{R}^{n(n+1)/2}$ by $f_{i,j}(A) = b_{ij}$, $j \geq i$ (we may call coordinates of $\mathbb{R}^{n(n+1)/2}$ by x_{ij}, $i \leq j$ for convenience). Similarly to Exercise 2.2(b), one can check that the value $(x_{ii} = 1, x_{ij} = 0)$ is regular, so the group $O_2(\mathbb{R})$ is a smooth manifold, and then use Exercise 2.2(a).

2.4. $SL_n(\mathbb{R})$ is a smooth manifold

(a) Let $f : \mathbb{R}^k \to \mathbb{R}$ be a homogeneous polynomial of degree $m \geq 1$. Prove *Euler's relation*

$$\langle \text{grad } f(x), x \rangle = m f(x),$$

where

$$\text{grad } f(x) = \left(\frac{\partial f}{\partial x_1}(x), \frac{\partial f}{\partial x_2}(x), \dots, \frac{\partial f}{\partial x_k}(x) \right).$$

Hint: Differentiate $\lambda \mapsto f(\lambda x_1, \lambda x_2, \dots, \lambda x_k)$ with respect to λ and use homogeneity.

(b) Let $f : \mathbb{R}^k \to \mathbb{R}$ be a homogeneous polynomial of degree $m \geq 1$. Show that every value $y \neq 0$ is a regular value of f.

(c) Use the fact that $\det A$ is a homogeneous polynomial in the entries of A in order to show that the special linear group $SL_n(\mathbb{R}) = \{A \in M_n(\mathbb{R}) \,|\, \det A = 1\}$ is a smooth manifold.

Solution:

(a) Let $g(\lambda) = f(\lambda x_1, \dots, \lambda x_k) = \lambda^m f(x_1, \dots, x_k)$. Using the chain rule, we obtain

$$g'(\lambda) = \sum_{i=1}^{k} \frac{\partial f}{\partial x_i}(\lambda x) \frac{\mathrm{d}(\lambda x_i)}{\mathrm{d}\lambda} = \sum_{i=1}^{k} \frac{\partial f}{\partial x_i}(\lambda x) x_i.$$

On the other hand,

$$g'(\lambda) = m\lambda^{m-1} f(x_1, \dots, x_k).$$

Choosing $\lambda = 1$, we obtain

$$mf(x) = \sum_{i=1}^{k} \frac{\partial f}{\partial x_i}(x) x_i = \langle \operatorname{grad} f(x), x \rangle$$

(b) Let f be a homogeneous polynomial of degree $m \geq 1$, and let $y \neq 0$. Let $x \in f^{-1}(y)$. Using (a), we have

$$\langle \operatorname{grad} f(x), x \rangle = mf(x) = my \neq 0.$$

This implies that $\operatorname{grad} f(x) \neq 0$, so $Df(x) : \mathbb{R}^k \to \mathbb{R}$ is surjective for all $x \in f^{-1}(y)$. Therefore, $y \neq 0$ is a regular value.

(c) The group $SL_n(\mathbb{R}) \subset M_n(\mathbb{R}) = \mathbb{R}^{n^2}$ is exactly $f^{-1}(1)$, where $f(A) = \det A$. Now, f is a homogeneous polynomial of degree n in $\mathbb{R}^{n^2}$, so 1 is a regular value of f by (b). Thus, $SL_n(\mathbb{R})$ is a smooth manifold of dimension $n^2 - 1$.

2.5. (a) Show that a directional derivative is a derivation (i.e. check the Leibniz rule).

(b) Show that derivations form a vector space.

Solution:

Both proofs are straightforward:

(a) If $\gamma : (-\varepsilon, \varepsilon) \to M$, $\gamma(0) = p$, and $f, g \in C^\infty(M, p)$, then

$$\begin{aligned}\gamma'(0)(fg) &= (fg \circ \gamma)'(0) = ((f \circ \gamma)(g \circ \gamma))'(0) = \\ &= (f \circ \gamma)'(0)(g \circ \gamma)(0) + (f \circ \gamma)(0)(g \circ \gamma)'(0) = \gamma'(0)(f)g(p) + f(p)\gamma'(0)(g)\end{aligned}$$

(b) Let $\delta_1, \delta_2 \in D^\infty(M, p)$, $f \in C^\infty(M, p)$, $\lambda_1, \lambda_2 \in \mathbb{R}$. Define

$$(\lambda_1\delta_1 + \lambda_2\delta_2)(f) = \lambda_1\delta_1(f) + \lambda_2\delta_2(f)$$

Then, if $g \in C^\infty(M, p)$, we have

$$\begin{aligned}(\lambda_1\delta_1 + \lambda_2\delta_2)(fg) &= \lambda_1\delta_1(fg) + \lambda_2\delta_2(fg) = \\ &= \lambda_1\delta_1(f)g(p) + \lambda_1 f(p)\delta_1(g) + \lambda_2\delta_2(f)g(p) + \lambda_2 f(p)\delta_2(g) = \\ &= (\lambda_1\delta_1 + \lambda_2\delta_2)(f)g(p) + f(p)(\lambda_1\delta_1 + \lambda_2\delta_2)(g),\end{aligned}$$

so any linear combination of derivations is also a derivation. All the other parts of the definition of a vector space can be easily verified.

2.6. ($\star$) Let M be the group $GL_n(\mathbb{R})$. Define a curve $\gamma : \mathbb{R} \to M$ by $\gamma(t) = I(1+t)$. Let $f : M \to \mathbb{R}$ be a function defined by $f(A) = \det A$. Compute $\gamma'(0)(f)$.

Solution:

$$\gamma'(0)(f) = (f \circ \gamma)'(0) = (\det{(I(1+t))})'(0) = ((1+t)^n)'(0) = n(1+t)^{n-1}|_{t=0} = n$$