Durham University
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Michaelmas 2014

Riemannian Geometry IV, Homework 2 (Week 2)

Due date for starred problems: **Wednesday, October 22**.

2.1. (a) Let U be an open subset of $\mathbb{R}^n$. Show that U is a smooth manifold.

(b) Show that the general linear group $GL_n(\mathbb{R}) = \{A \in M_n(\mathbb{R}) \,|\, \det A \neq 0\}$ is a smooth manifold.

2.2. ($\star$)

(a) Show that the set of $n \times n$ matices real matrices with positive determinant is an open subset of $M_n(\mathbb{R})$.

(b) Show that the special orthogonal group $SO_2(\mathbb{R}) = \{A \in M_2(\mathbb{R}) \,|\, A^t A = I,\ \det A = 1\}$ is a smooth 1-manifold.

2.3. Show that the special orthogonal group $SO_n(\mathbb{R}) = \{A \in M_n(\mathbb{R}) \,|\, A^t A = I,\ \det A = 1\}$ is a smooth manifold.

2.4. $SL_n(\mathbb{R})$ **is a smooth manifold**

(a) Let $f : \mathbb{R}^k \to \mathbb{R}$ be a homogeneous polynomial of degree $m \geq 1$. Prove *Euler's relation*

$$\langle \operatorname{grad} f(x), x \rangle = m f(x),$$

where

$$\operatorname{grad} f(x) = \left(\frac{\partial f}{\partial x_1}(x), \frac{\partial f}{\partial x_2}(x), \ldots, \frac{\partial f}{\partial x_k}(x) \right).$$

Hint: Differentiate $\lambda \mapsto f(\lambda x_1, \lambda x_2, \ldots, \lambda x_k)$ with respect to λ and use homogeneity.

(b) Let $f : \mathbb{R}^k \to \mathbb{R}$ be a homogeneous polynomial of degree $m \geq 1$. Show that every value $y \neq 0$ is a regular value of f.

(c) Use the fact that $\det A$ is a homogeneous polynomial in the entries of A in order to show that the special linear group $SL_n(\mathbb{R}) = \{A \in M_n(\mathbb{R}) \,|\, \det A = 1\}$ is a smooth manifold.

2.5. (a) Show that a directional derivative is a derivation (i.e. check the Leibniz rule).

(b) Show that derivations form a vector space.

2.6. ($\star$) Let M be the group $GL_n(\mathbb{R})$. Define a curve $\gamma : \mathbb{R} \to M$ by $\gamma(t) = I(1 + t)$. Let $f : M \to \mathbb{R}$ be a function defined by $f(A) = \det A$. Compute $\gamma'(0)(f)$.