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Riemannian Geometry IV, Solutions 4 (Week 4)

4.1. Let M and N be smooth manifolds. Using local coordinates, explain why $T_{(p,q)}(M\times N) = T_pM\oplus T_qN$ for $p \in M$ and $q \in N$.

Solution:

If (U_i, V_i, φ_i), $i \in I$, is an atlas for M, and (U_j, V_j, ψ_j), $j \in J$, is an atlas for N, then we get an atlas for $M \times N$ by taking products $(U_i \times U_j, V_i \times V_j, \varphi_i \times \psi_j)$ for $(i, j) \in I \times J$. It follows that if $(x_1, \dots, x_m)$ and $(y_1, \dots, y_n)$ are local coordinates at $p \in M$ and $q \in N$, then $(x_1, \dots, x_m, y_1, \dots, y_n)$ are local coordinates at $(p, q) \in M \times N$. Hence we see that

$$\begin{aligned} T_{(p,q)}(M \times N) &= \langle\frac{\partial}{\partial x_1}, \dots, \frac{\partial}{\partial x_m}, \frac{\partial}{\partial y_1}, \dots, \frac{\partial}{\partial y_n}\rangle \\ &= \langle\frac{\partial}{\partial x_1}, \dots, \frac{\partial}{\partial x_m}\rangle \oplus \langle\frac{\partial}{\partial y_1}, \dots, \frac{\partial}{\partial y_n}\rangle = T_pM \oplus T_qN. \end{aligned}$$

4.2. Let $M \subset \mathbb{R}^n$ be a smooth manifold given by the equation $f(x_1, \dots, x_n) = a$. Let $p \in M$ and $v \in T_pM$. Show that the vector $v = (v_1, \dots, v_n)$ satisfies the equation

$$\sum_{i=1}^{n} \frac{\partial f}{\partial x_i} v_i = 0,$$

or, equivalently, $\langle \text{grad } f(p), v\rangle = 0$.

Solution:

Let $\gamma(t)$ be a curve on M such that $\gamma(0) = p$ and $v = \gamma'(0)$. Then we have $f(\gamma_1(t), \dots, \gamma_n(t)) = a$. We compute $\frac{df(\gamma(t))}{dt}\big|_{t=0}$ using the chain rule:

$$\frac{df(\gamma(t))}{dt}\Big|_{t=0} = \sum_{i=1}^{n} \frac{\partial f}{\partial x_i}\frac{d\gamma_i(t)}{dt}\Big|_{t=0} = \sum_{i=1}^{n} \frac{\partial f}{\partial x_i} v_i$$

Notice that as $f(\gamma(t)) \equiv a$, we also have $\frac{df(\gamma(t))}{dt}\big|_{t=0} = 0$ which implies

$$\sum_{i=1}^{n} \frac{\partial f}{\partial x_i} = 0.$$

4.3. $(\star)$ Let X be a vector field on $\mathbb{R}^3$ defined by

$$X(x, y, z) = y\frac{\partial}{\partial x} - x\frac{\partial}{\partial y} + (x + y + z)\frac{\partial}{\partial z}$$

Let $M \subset \mathbb{R}^3$ be a cylinder $\{(x, y, z) \in \mathbb{R}^3 \mid x^2 + y^2 = 1\}$.

(a) Show that $X \in \mathfrak{X}(M)$.

(b) Express X in terms of $\dfrac{\partial}{\partial\varphi}$ and $\dfrac{\partial}{\partial h}$, where (φ, h) are cylindrical coordinates on M, i.e.

$$(x, y, z) = (\cos\varphi, \sin\varphi, h)$$

Solution:

(a) Identifying $T_{(x,y,z)}\mathbb{R}^3$ with $\mathbb{R}^3$, we can write that $X(x,y,z) = (y, -x, x+y+z) \in \mathbb{R}^3$. The cylinder M is a level set of a function $f(x,y,z) = x^2 + y^2$, thus, according to Exercise 4.2, vector $X(x,y,z)$ is tangent to the cylinder if and only if $\langle \text{grad } f(x,y,z), X(x,y,z)\rangle = 0$. Since grad $f(x,y,z) = (2x, 2y, 0)$, we see that

$$\langle \text{grad } f(x,y,z), X(x,y,z)\rangle = \langle (2x,2y,0), (y,-x,x+y+z)\rangle = 0$$

(b) Since $(x,y,z) = (\cos\varphi, \sin\varphi, h)$, we have

$$\begin{aligned}
\frac{\partial}{\partial\varphi} &= (-\sin\varphi, \cos\varphi, 0) = -\sin\varphi\frac{\partial}{\partial x} + \cos\varphi\frac{\partial}{\partial y}, \\
\frac{\partial}{\partial h} &= (0,0,1) = \frac{\partial}{\partial z}, \\
X(x,y,z) &= (\sin\varphi, -\cos\varphi, \cos\varphi + \sin\varphi + h).
\end{aligned}$$

Thus,

$$X(\varphi, h) = -\frac{\partial}{\partial\varphi} + (\cos\varphi + \sin\varphi + h)\frac{\partial}{\partial h}$$

4.4. (a) ($\star$) Find vector fields $X, Y \in \mathfrak{X}(\mathbb{T}^2)$ such that $\{X(p), Y(p)\}$ is a basis for $T_p\mathbb{T}^2$ for all $p \in \mathbb{T}^2$.
Hint: you may embed the torus $\mathbb{T}^2$ into $\mathbb{R}^3$ as a surface of revolution.

(b) Find vector fields $X, Y, Z \in \mathfrak{X}(S^3)$ such that $\{X(p), Y(p), Z(p)\}$ is a basis for T_pS^3 for all $p \in S^3$.
Hint: you may use the embedding of S^3 described in Exercise 3.3.

Solution:

(a) We may embed the torus into $\mathbb{R}^3$ as

$$(x,y,z) = ((\cos\vartheta + 2)\cos\varphi, (\cos\vartheta + 2)\sin\varphi, \sin\vartheta),$$

where $\vartheta, \varphi \in [0, 2\pi)$. The curves $\vartheta = \text{const}$ are *parallels* of the torus, the curves $\varphi = \text{const}$ are *meridians*. It is easy to see that at every point of $\mathbb{T}^2$ the meridian is orthogonal to the parallel, so tangent vectors to them compose a basis of the tangent space. Thus, it is sufficient to find two non-vanishing vector fields X and Y, where X is tangent to meridians, and Y is tangent to parallels. We may define

$$\begin{aligned}
X(\vartheta,\varphi) &= \frac{\partial}{\partial\vartheta} = (-\sin\vartheta\cos\varphi, -\sin\vartheta\sin\varphi, \cos\vartheta), \\
Y(\vartheta,\varphi) &= \frac{\partial}{\partial\varphi} = (-(\cos\vartheta+2)\sin\varphi, (\cos\vartheta+2)\cos\varphi, 0).
\end{aligned}$$

The fields X and Y can also be written in terms of (x,y,z)-coordinates in $\mathbb{R}^3$:

$$\begin{aligned}
X(x,y,z) &= \left(\frac{-xz}{\sqrt{x^2+y^2}}, \frac{-yz}{\sqrt{x^2+y^2}}, \sqrt{1-z^2}\ \text{sgn}\,(x^2+y^2-4)\right), \\
Y(x,y,z) &= (-y, x, 0).
\end{aligned}$$

(b) Embedding S^3 as the unit sphere into $\mathbb{R}^4$, we see that choosing the vector

$$(-y, x, -w, z) \in T_{(x,y,z,w)}S^3 \subset T_{(x,y,z,w)}\mathbb{R}^4$$

describes a nowhere-vanishing vector field (you should check that this vector actually lies in $T_{(x,y,z,w)}S^3$ by checking that it is normal to the normal direction to S^3 at this point).

Permuting coordinates in an appropriate way, you then should be able to find (up to an overall multiplication by ± 1) five other nowhere-vanishing vector fields. Choosing carefully three of these, you should then verify that the vectors at each point of S^3 compose a basis of the tangent space. You can do this by checking that they are linearly independent.