Durham University
Pavel Tumarkin Michaelmas 2014

Riemannian Geometry IV, Term 1 (Sections 1–2)

1 Smooth manifolds

"Smooth" means "infinitely differentiable", C^∞.

Definition 1.1. Let M be a set. An n-dimensional smooth atlas on M is a collection of triples $(U_\alpha, V_\alpha, \varphi_\alpha)$, where $\alpha \in I$ for some indexing set I, s.t.

(a) $U_\alpha \subseteq M$; $V_\alpha \subseteq \mathbb{R}^n$ is open $\forall \alpha \in I$;

(b) $\bigcup_{\alpha \in I} U_\alpha = M$;

(c) Each $\varphi_\alpha : U_\alpha \to V_\alpha$ is a bijection;

(d) For every $\alpha, \beta \in I$ such that $U_\alpha \cap U_\beta \neq \emptyset$ the composition $\varphi_\beta \circ \varphi_\alpha^{-1}\big|_{\varphi_\alpha(U_\alpha \cap U_\beta)} : \varphi_\alpha(U_\alpha \cap U_\beta) \to \varphi_\beta(U_\alpha \cap U_\beta)$ is a smooth map for all ordered pairs (α, β), where $\alpha, \beta \in I$.

The number n is called the dimension of M, the maps φ_α are called coordinate charts, the compositions $\varphi_\beta \circ \varphi_\alpha^{-1}$ are called transition maps or changes of coordinates.

Example 1.2. Two atlases on a circle $S^1 \subset \mathbb{R}^2$.

Definition 1.3. Let M have a smooth atlas. A set $A \subseteq M$ is open if for every $\alpha \in I$ the set $\varphi_\alpha(A \cap U_\alpha)$ is open in $\mathbb{R}^n$. If $A \subset M$ is open and $x \in A$, A is called an open neighborhood of x.

Definition 1.4. M is called Hausdorff if for each $x, y \in M$, $x \neq y$, there exist open sets $A_x \ni x$ and $A_y \ni y$ such that $A_x \cap A_y = \emptyset$.

Example 1.5. An example of a non-Hausdorff space: a line with a double point.

Definition 1.6. M is called a smooth n-dimensional manifold if M has a countable n-dimensional smooth atlas and M is Hausdorff

Example 1.7. Atlas for a square in $\mathbb{R}^2$.

Example. Examples of smooth manifolds: torus, Klein bottle, 3-torus, real projective space.

Definition 1.8. Let $U \subseteq \mathbb{R}^n$ be open, $m < n$, and let $f : U \to \mathbb{R}^m$ be a smooth map (i.e., all the partial derivatives are smooth). Let $Df(x) = (\frac{\partial f_i}{\partial x_j})$ be the matrix of partial derivatives at $x \in U$ (differential or Jacobi matrix). Then

(a) $x \in \mathbb{R}^n$ is a regular point of f if rk $Df(x) = m$ (i.e., $Df(x)$ has a maximal rank);

(b) $y \in \mathbb{R}^m$ is a regular value of f if the full preimage $f^{-1}(y)$ consists of regular points only.

Theorem 1.9 (Corollary of Implicit Function Theorem)**.** *Let $U \subset \mathbb{R}^n$ be open, $f : U \to \mathbb{R}^m$ smooth, $m < n$. If $y \in f(U)$ is a regular value of f then $f^{-1}(y) \subset U \subset \mathbb{R}^n$ is an $(n-m)$-dimensional smooth manifold.*

Examples 1.10–1.11. An ellipsoid as a smooth manifold; matrix groups are smooth manifolds.

2 Tangent space

Definition 2.1. Let $f : M^m \to N^n$ be a map of smooth manifolds with atlases $(U_i, \varphi_i(U_i), \varphi_i)_{i\in I}$ and $(W_j, \psi_j(W_j), \psi_j)_{j\in J}$. The map f is smooth if it induces smooth maps between open sets in $\mathbb{R}^m$ and $\mathbb{R}^n$, i.e. if $\psi_j \circ f \circ \varphi_i^{-1}\big|_{\varphi_i(U_i\cap f^{-1}(W_j\cap f(U_i)))}$ is smooth for all $i \in I$, $j \in J$.

If f is a bijection and both f and f^{-1} are smooth then f is called a diffeomorphism.

Definition 2.2. A derivation on the set $C^\infty(M,p)$ of all smooth functions on M defined in a neighborhood of p is a linear map $\delta : C^\infty(M,p) \to \mathbb{R}$, s.t. for all $f, g \in C^\infty(M,p)$ holds $\delta(f \cdot g) = f(p)\delta(g) + \delta(f)g(p)$ (the Leibniz rule).

The set of all derivations is denoted by $\mathcal{D}^\infty(M,p)$. This is a real vector space (exercise).

Definition 2.3. The space $\mathcal{D}^\infty(M,p)$ is called the tangent space of M at p, denoted T_pM. Derivations are tangent vectors.

Definition 2.4. Let $\gamma : (a,b) \to M$ be a smooth curve in M, $t_0 \in (a,b)$, $\gamma(t_0) = p$ and $f \in C^\infty(M,p)$. Define the directional derivative $\gamma'(t_0)(f) \in \mathbb{R}$ of f at p along γ by

$$\gamma'(t_0)(f) = \lim_{s\to 0} \frac{f(\gamma(t_0+s)) - f(\gamma(t_0))}{s} = (f\circ\gamma)'(t_0) = \frac{d}{dt}\Big|_{t=t_0} (f\circ\gamma)$$

Directional derivatives are derivations (exercise).

Remark. Two curves γ_1 and γ_2 through p may define the same directional derivative.

Notation. Let M^n be a manifold, $\varphi : U \to V \subseteq \mathbb{R}^n$ a chart at $p \in U \subset M$. For $i = 1, \ldots, n$ define the curves $\gamma_i(t) = \varphi^{-1}(\varphi(p) + e_i t)$ for small $t > 0$ (here $\{e_i\}$ is a basis of $\mathbb{R}^n$).

Definition 2.5. Define $\frac{\partial}{\partial x_i}\big|_p = \gamma_i'(0)$, i.e.

$$\frac{\partial}{\partial x_i}\Big|_p (f) = (f\circ\gamma_i)'(0) = \frac{d}{dt}(f\circ\varphi^{-1})(\varphi(p)+te_i)\big|_{t=0} = \frac{\partial}{\partial x_i}(f\circ\varphi^{-1})(\varphi(p)),$$

where $\frac{\partial}{\partial x_i}$ on the right is just a classical partial derivative.

By definition, we have

$$\langle \frac{\partial}{\partial x_1}, \ldots, \frac{\partial}{\partial x_n} \rangle \subseteq \{\text{Directional derivatives}\} \subseteq \mathcal{D}^\infty(M,p)$$

Proposition 2.6. $\langle \frac{\partial}{\partial x_1}, \ldots, \frac{\partial}{\partial x_n} \rangle = \{$*Directional derivatives*$\} = \mathcal{D}^\infty(M,p)$.

Lemma 2.7. *Let $\varphi : U \subseteq M \to \mathbb{R}^n$ be a chart, $\varphi(p) = 0$. Let $\tilde{\gamma}(t) = (\sum_{i=1}^n k_i e_i)\, t : \mathbb{R} \to \mathbb{R}^n$ be a line, where $\{e_1, \ldots, e_n\}$ is a basis, $k_i \in \mathbb{R}$. Define $\gamma(t) = \varphi^{-1}\circ\tilde{\gamma}(t) \in M$. Then $\gamma'(0) = \sum_{i=1}^n k_i \frac{\partial}{\partial x_i}$.*

Example 2.8. For the group $SL_n(\mathbb{R}) = \{A \in M_n \mid \det A = 1\}$, the tangent space at I is the set of all trace-free matrices: $T_I(SL_n(\mathbb{R})) = \{X \in M_n(\mathbb{R}) \mid \operatorname{tr} X = 0\}$.

Proposition 2.9. *(Change of basis for T_pM). Let M^n be a smooth manifold, $\varphi_\alpha : U_\alpha \to V_\alpha$ a chart, $(x_1^\alpha, \ldots, x_n^\alpha)$ the coordinates in V_α. Let $p \in U_\alpha \cap U_\beta$. Then $\frac{\partial}{\partial x_j^\beta}\Big|_p = \sum_{i=1}^n \frac{\partial x_i^\beta}{\partial x_j^\alpha}\frac{\partial}{\partial x_i^\alpha}$, where $\frac{\partial x_i^\beta}{\partial x_j^\alpha} = \frac{\partial(\varphi_\beta^i\circ\varphi_\alpha^{-1})}{\partial x_j^\alpha}(\varphi(p))$, $\varphi_\beta^i = \pi_i\circ\varphi_\beta$.*

Definition 2.10. Let M, N be smooth manifolds, let $f : M \to N$ be a smooth map. Define a linear map $Df(p) : T_pM \to T_{f(p)}N$ called the <u>differential</u> of f at p by $Df(p)\gamma'(0) = (f \circ \gamma)'(0)$ for a smooth curve $\gamma \in M$ with $\gamma(0) = p$.

Remark. $Df(p)$ is well defined (exercise).

Lemma 2.11. *(a) If φ is a chart, then $D\varphi(p) : T_pM \to T_{\varphi(p)}\mathbb{R}^n$ is the identity map taking $\frac{\partial}{\partial x_i}\Big|_p$ to $\frac{\partial}{\partial x_i}$*

(b) For $M \xrightarrow{f} N \xrightarrow{g} L$ holds $D(g \circ f)(p) = Dg(f(p)) \circ Df(p)$.

Example 2.12. Differential of a map from a disc to a sphere.

Tangent bundle and vector fields

Definition 2.13. Let M be a smooth manifold. A disjoint union $TM = \cup_{p\in M} T_pM$ of tangent spaces to each $p \in M$ is called a <u>tangent bundle</u>.

There is a canonical projection $\Pi : TM \to M$, $\Pi(v) = p$ for every $v \in T_pM$.

Proposition 2.14. *The tangent bundle TM has a structure of $2n$-dimensional smooth manifold, s.t. $\Pi : TM \to M$ is a smooth map.*

Definition 2.15. A <u>vector field</u> X on a smooth manifold M is a smooth map $X : M \to TM$ such that $\forall p \in M$ $X(p) \in T_pM$

The set of all vector fields on M is denoted by $\mathfrak{X}(M)$.

Remark 2.16. (a) $\mathfrak{X}(M)$ has a structure of a vector space.

(b) Vector fields can be multiplied by smooth functions.

(c) Taking a coordinate chart $(U, \varphi = (x_1, \dots, x_n))$, any vector field X can be written in U as $X(p) = \sum_{i=1}^n f_i(p)\frac{\partial}{\partial x_i} \in T_pM$, where $\{f_i\}$ are some smooth functions on U.

Examples 2.17–2.18. Vector fields on $\mathbb{R}^2$ and 2-sphere.

Remark 2.19. Observe that for $X = \sum a_i(p)\frac{\partial}{\partial x_i} \in \mathfrak{X}(M)$ we have $X(p) \in T_pM$, i.e. $X(p)$ is a directional derivative at $p \in M$. Thus, we can use the vector field to differentiate a function $f \in C^\infty(M)$ by $(Xf)(p) = \sum a_i(p)\frac{\partial f}{\partial x_i}\Big|_p$, so that we get another smooth function $Xf \in C^\infty(M)$.

Proposition 2.20. *Let $X, Y \in \mathfrak{X}(M)$. Then there exists a unique vector field $Z \in \mathfrak{X}(M)$ such that $Z(f) = X(Y(f)) - Y(X(f))$ for all $f \in C^\infty(M)$.*

This vector field $Z = XY - YX$ is denoted by $[X, Y]$ and called the <u>Lie bracket</u> of X and Y.

Proposition 2.21. *Properties of Lie bracket:*

(a) $[X, Y] = -[Y, X]$;

(b) $[aX + bY, Z] = a[X, Z] + b[Y, Z]$ for $a, b \in \mathbb{R}$;

(c) $[[X, Y], Z] + [[Y, Z], X] + [[Z, X], Y] = 0$ (<u>Jacobi identity</u>);

(d) $[fX, gY] = fg[X, Y] + f(Xg)Y - g(Yf)X$ for $f, g \in C^\infty(M)$.

Definition 2.22. A Lie algebra is a vector space $\mathfrak{g}$ with a binary operation $[\cdot,\cdot] : \mathfrak{g} \times \mathfrak{g} \to \mathfrak{g}$ called the Lie bracket which satisfies first three properties from Proposition 2.21.

Proposition 2.21 implies that $\mathfrak{X}(M)$ is a Lie algebra.

Theorem 2.23 (The Hairy Ball Theorem)**.** *There is no non-vanishing continuous vector field on an even-dimensional sphere* S^{2m}*.*