Durham University
Pavel Tumarkin

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Riemannian Geometry IV, Term 1 (Section 5)

5 Geodesics

5.1 Geodesics as solutions of ODE's

Definition 5.1. Given (M,g), a curve $c:[a,b]\to M$ is a geodesic if $\frac{D}{dt}c'(t)=0$ for all $t\in[a,b]$ (i.e., $c'(t)\in\mathfrak{X}_c(M)$ is parallel along c).

Lemma 5.2. *If c is a geodesic then c is parametrized proportionally to the arc length.*

Theorem 5.3. *Given a Riemannian manifold (M,g), $p\in M$, $v\in T_pM$, there exists $\varepsilon>0$ and a unique geodesic $c:(-\varepsilon,\varepsilon)\to M$ such that $c(0)=p$, $c'(0)=v$.*

Examples 5.4–5.5 Geodesics in Euclidean space, on a sphere, and in the upper half-plane model $\mathbb{H}^2$.

5.2 Geodesics as distance-minimizing curves. First variation formula of the length

Definition 5.6. Let $c:[a,b]\to M$ be a smooth curve. A smooth map $F:(-\varepsilon,\varepsilon)\times[a,b]\to M$ is a (smooth) variation of c if $F(0,t)=c(t)$. Variation is proper if $F(s,a)=c(a)$ and $F(s,b)=c(b)$ for all $s\in(-\varepsilon,\varepsilon)$.

Variation can be considered as a family of the curves $F_s(t)=F(s,t)$.

Definition 5.7. A variational vector field $X\in\mathfrak{X}_c(M)$ of a variation F is defined by $X(t)=\frac{\partial F}{\partial s}(0,t)$.

Definition 5.8. The length $l:(-\varepsilon,\varepsilon)\to[0,\infty)$ and energy $E:(-\varepsilon,\varepsilon)\to[0,\infty)$ of a variation F are defined by

$$l(s)=\int_a^b \|\frac{\partial F}{\partial t}(s,t)\|\,\mathrm{d}t, \qquad E(s)=\frac{1}{2}\int_a^b \|\frac{\partial F}{\partial t}(s,t)\|^2\,\mathrm{d}t$$

Remark. $l(s)$ is the length of the curve $F_s(t)$.

Theorem 5.9. *A smooth curve c is a geodesic if and only if c is parametrized proportionally to the arc length and $l'(0)=0$ for every proper variation of c.*

Corollary 5.10. *Let $c:[a,b]\to M$ be the shortest curve from $c(a)$ to $c(b)$, and c is parametrized proportionally to the arc length. Then c is geodesic.*

Remark. The converse is false (e.g., on the sphere).

Lemma 5.11 (Symmetry Lemma). *Let $W\subset\mathbb{R}^2$ be an open set and $F:W\to M$, $(s,t)\mapsto F(s,t)$, be a smooth map. Let $\frac{D}{dt}$ be the covariant derivative along $F_s(t)$ and $\frac{D}{ds}$ be the covariant derivative along $F_t(s)$. Then $\frac{D}{dt}\frac{\partial F}{\partial s}=\frac{D}{ds}\frac{\partial F}{\partial t}$.*

Theorem 5.12 (First variation formula of length)**.** *Let $F : (-\varepsilon, \varepsilon) \times [a, b] \to M$ be a variation of a smooth curve $c(t)$, $c'(t) \neq 0$. Let $X(t)$ be its variational vector field and $l : (-\varepsilon, \varepsilon) \to [0, \infty)$ its length. Then*

$$l'(0) = \int_a^b \frac{1}{\|c'(t)\|} \frac{d}{dt} \langle X(t), c'(t) \rangle \, \mathrm{d}t - \int_a^b \frac{1}{\|c'(t)\|} \langle X(t), \frac{D}{dt} c'(t) \rangle \, \mathrm{d}t$$

Corollary 5.13. *(a) If $c(t)$ is parametrized proportionally to the arc length, $\|c'(t)\| \equiv c$, then* $l'(0) = \frac{1}{c} \langle X(b), c'(b) \rangle - \frac{1}{c} \langle X(a), c'(a) \rangle - \frac{1}{c} \int_a^b \langle X(t), \frac{D}{dt} c'(t) \rangle \, \mathrm{d}t$*;*

(b) if $c(t)$ is geodesic, then $l'(0) = \frac{1}{c} \langle X(b), c'(b) \rangle - \frac{1}{c} \langle X(a), c'(a) \rangle$;

(c) if F is proper and c is parametrized proportionally to the arc length, then $l'(0) = -\frac{1}{c} \int_a^b \langle X(t), \frac{D}{dt} c'(t) \rangle \, \mathrm{d}t$;

(d) if F is proper and c is geodesic, then $l'(0) = 0$.

Lemma 5.14. *Any vector field X along $c(t)$ with $X(a) = X(b) = 0$ is a variational vector field for some proper variation F.*

5.3 Exponential map and Gauss Lemma

Let $p \in M$, $v \in T_pM$. Denote by $c_v(t)$ the unique maximal geodesic (i.e., the domain is maximal) with $c_v(0) = p$, $c_v'(0) = v$.

Definition 5.15. If $c_v(1)$ exists, define $\exp_p : T_pM \to M$ by $\exp_p(v) = c_v(1)$, the exponential map at p.

Example 5.16. Exponential map on the sphere S^2: length of c_v from p to $c_v(1)$ equals $\|v\|$.

Notation. $B_r(0_p) = \{v \in T_pM \mid \|v\| < r\} \subset T_pM$ is a ball of radius r centered at 0_p.

Proposition 5.17. *(without proof)*
For any $p \in (M, g)$ there exists $r > 0$ such that $\exp_p : B_r(0_p) \to \exp_p(B_r(0_p))$ is a diffeomorphism.

Example. On S^2 the set $\exp_p(B_{\pi/2}(0_p))$ is a hemisphere, so that every geodesic starting from p is orthogonal to the boundary of this set.

Theorem 5.18 (Gauss Lemma)**.** *Let (M, g) be a Riemannian manifold, $p \in M$, and let $\varepsilon > 0$ be such that $\exp_p : B_\varepsilon(0_p) \to \exp_p(B_\varepsilon(0_p))$ is a diffeomorphism. Define $A_\delta = \{\exp_p(w) \mid \|w\| = \delta\}$ for every $0 < \delta < \varepsilon$. Then every radial geodesic $c : t \mapsto \exp_p(tv)$, $t \geq 0$, is orthogonal to A_δ.*

Remark 5.19. The curve $c_v(t) = \exp_p(tv)$ is indeed geodesic; every geodesic γ through p can be written as $\gamma(t) = \exp_p(tw)$ for appropriate $w \in T_pM$.

Definition. Denote $B_\varepsilon(p) = \exp_p(B_\varepsilon(0_p)) \subset M$, a geodesic ball.

Lemma 5.20. *Let (M, g) be a Riemannian manifold and $p \in M$. Let $\varepsilon > 0$ be small enough such that $\exp_p : B_\varepsilon(0_p) \to B_\varepsilon(p) \subset M$ is a diffeomorphism. Let $\gamma : [0, 1] \to B_\varepsilon(p) \setminus \{p\}$ be any curve. Then there exists a curve $v : [0, 1] \to T_pM$, $\|v(s)\| = 1$ for all $s \in [0, 1]$, and a positive function $r : [0, 1] \to \mathbb{R}_+$, such that $\gamma(s) = \exp_p(r(s)v(s))$.*

Lemma 5.21. *Let $r : [0,1] \to \mathbb{R}_+$, $v : [0,1] \to S_pM = \{w \in T_pM \mid \|w\| = 1\}$. Define $\gamma : [0,1] \to B_\varepsilon(p) \setminus \{p\}$ by $\gamma(s) = \exp_p(r(s)v(s))$. Then the length $l(\gamma) \geq |r(1) - r(0)|$, and the equality holds if and only if γ is a reparametrization of a radial geodesic (i.e. $v(s) \equiv \|v(0)\|$ and $r(s)$ is a strictly increasing or decreasing function).*

Corollary 5.22. *Given a point $p \in M$, there exists $\varepsilon > 0$ such that for any $q \in B_\varepsilon(p)$ there exists a curve $c(t)$ connecting p and q and satisfying $l(c) = d(p,q)$. (This curve is a radial geodesic).*

Remark. According to Corollary 5.22, there is $\varepsilon > 0$ such that $B_\varepsilon(p)$ coincides with ε-ball at p, i.e. with $\{q \in M \mid d(p,q) < \varepsilon\}$.

Proposition 5.23. *(without proof)*
Let $p \in M$. Then there is an open neighborhood U of p and $\varepsilon > 0$ such that $\forall\, q \in U$ $\exp_q : B_\varepsilon(0_q) \to B_\varepsilon(q)$ is a diffeomorphism.

5.4 Hopf-Rinow Theorem

Definition 5.24. A geodesic $c : [a,b] \to M$ is minimal if $l(c) = d(c(a), c(b))$. A geodesic $c : \mathbb{R} \to M$ is minimal if its restriction $c|_{[a,b]}$ is minimal for each segment $[a,b] \subset \mathbb{R}$.

Example. No minimal geodesics in S^2, all geodesics in $\mathbb{E}^2$ are minimal.

Definition 5.25. A Riemannian manifold (M,g) is geodesically complete if every geodesic $c : [a,b] \to M$ can be extended to a geodesic $\tilde{c} : \mathbb{R} \to M$ (i.e. can be extended infinitely in both directions). Equivalently, $\exp_p$ is defined on the whole T_pM for all $p \in M$.

Theorem 5.26 (Hopf-Rinow)**.** *Let (M,g) be a connected Riemannian manifold with distance function d. Then the following are equivalent:*

(a) (M,g) is complete (i.e. every Cauchy sequence converges);

(b) every closed and bounded subset is compact;

(c) (M,g) is geodesically complete.

Moreover, every of the conditions above implies

(d) for every $p, q \in M$ there exists a minimal geodesic connecting p and q.

Remark. A geodesic in (d) may not be unique. Further, (d) does not imply (c).

Remark. Theorem 5.26 uses the following notions defined in a metric space:

- $\{x_i\}$, $x_i \in M$, is a Cauchy sequence if $\forall \varepsilon > 0\, \exists N\, \forall m, n > N \;\; d(x_m, x_n) < \varepsilon$;
- a set $A \subset M$ is bounded if $A \subset B_r(p)$ for some $r > 0$, $p \in M$;
- a set $A \subset M$ is closed if $\{x_n \in A, x_n \to x\} \Rightarrow x \in A$;
- a set $A \subset M$ is compact if each open cover has a finite subcover;
- a set $A \subset M$ is sequentially compact if each sequence has a converging subsequence.

Some properties:
1. A compact set is sequentially compact, bounded, closed.
2. A compact metric space is complete.
3. In a complete metric space, a sequentially compact set is compact.