Durham University
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Differential Geometry III, Solutions 1 (Week 1)

1.3. ($\star$) An *epicycloid* $\boldsymbol{\alpha}$ is obtained as the locus of a point on the circumference of a circle of radius r which rolls without slipping on a circle of the same radius.

(a) Sketch $\boldsymbol{\alpha}$.

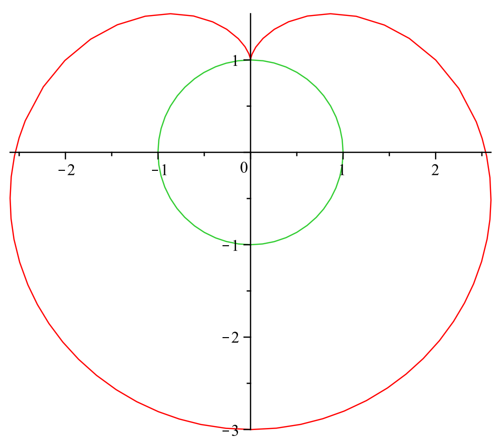
(b) Show that the epicycloid can be parametrized by

$$\boldsymbol{\alpha}(u) = (2r\sin u - r\sin 2u,\ 2r\cos u - r\cos 2u), \qquad u \in \mathbb{R}.$$

Find the length of $\boldsymbol{\alpha}$ between the singular points at $u = 0$ and $u = 2\pi$.

Solution:

The graph of the epicycloid is illustrated below for the value $r = 1$.



The inner (green) circle centered at $(0, 0)$ is fixed, and the second circle C rotates around it with a marked point on its perimeter tracing out the epicycloid. This point is at the bottom of the rotating circle at the moment $u = 0$ when the rotating circle is just on top of the fixed circle, i.e., at position $(0, r)$. As u increases, the center of C moves clockwise around the origin, and so does the point of contact between the fixed and the rotating circle, and also so does the marked point around the center of C in relation to the point of contact.

At the time u the center of the rotating circle C is located at $(2r\sin u, 2r\cos u)$. To this moment C has rotated clockwise around its moving center by a total length of $2ru$, where u is measured in radians. Therefore, the point of contact between the two circles, seen from the moving center of C, has moved clockwise by the angle u around its moving center, and the position of the point of contact relative to this moving center is $(r\sin(\pi + u), r\cos(\pi + u))$. The marked point has moved clockwise away from the point of contact by the same angle, and is therefore at position $(r\sin(\pi+2u), r\cos(\pi+2u))$ relative to the center of the moving circle. This means that the marked point lies at

$$(2r\sin u, 2r\cos u) + (r\sin(\pi+2u), r\cos(\pi+2u)) = (2r\sin u - r\sin 2u, 2r\cos u - r\cos 2u).$$

Now let

$$\boldsymbol{\alpha}(u) = (2r\sin u - r\sin 2u, 2r\cos u - r\cos 2u).$$

Then

$$\begin{aligned}
\boldsymbol{\alpha}'(u) &= 2r(\cos u - \cos 2u, -(\sin u - \sin 2u)) \\
\|\boldsymbol{\alpha}'(u)\|^2 &= 4r^2(2 - 2(\cos(-u)\cos(2u) - \sin(-u)\sin(2u)) \\
&= 4r^2(2 - 2\cos(2u-u)) = 4r^2(2 - 2\cos u) \\
&= 4r^2(2 - 2(\cos(u/2)\cos(u/2) - \sin(u/2)\sin(u/2))) \\
&= 16r^2\sin^2(u/2).
\end{aligned}$$

This implies that $\|\boldsymbol{\alpha}'(u)\| = 4r\sin(u/2)$ and

$$l(\boldsymbol{\alpha}) = \int_0^{2\pi} \|\boldsymbol{\alpha}'(u)\| du = 4r\int_0^{2\pi} \sin\frac{u}{2} du = 4r\left(-2\cos\frac{u}{2}\Big|_0^{2\pi}\right) = -8r(\cos\pi - \cos 0) = 16r.$$

1.4. ($\star$) (a) Let $\boldsymbol{\alpha}(u)$ and $\boldsymbol{\beta}(u)$ be two smooth plane curves. Show that

$$\frac{d}{du}(\boldsymbol{\alpha}(u)\cdot\boldsymbol{\beta}(u)) = \boldsymbol{\alpha}'(u)\cdot\boldsymbol{\beta}(u) + \boldsymbol{\alpha}(u)\cdot\boldsymbol{\beta}'(u),$$

where $\boldsymbol{\alpha}(u)\cdot\boldsymbol{\beta}(u)$ denotes a Euclidean dot product of vectors $\boldsymbol{\alpha}(u)$ and $\boldsymbol{\beta}(u)$.

(b) Let $\boldsymbol{\alpha}(u) : I \to \mathbb{R}^2$ be a smooth curve which does not pass through the origin. Suppose there exists $u_0 \in I$ such that the point $\boldsymbol{\alpha}(u_0)$ is the closest to the origin amongst all the points of the trace of $\boldsymbol{\alpha}$. Show that $\boldsymbol{\alpha}(u_0)$ is orthogonal to $\boldsymbol{\alpha}'(u_0)$.

Solution:

(a) Let $\boldsymbol{\alpha}(u) = (\alpha_1(u), \alpha_2(u))$, $\boldsymbol{\beta}(u) = (\beta_1(u), \beta_2(u))$. Then

$$\boldsymbol{\alpha}(u)\cdot\boldsymbol{\beta}(u) = \alpha_1(u)\beta_1(u) + \alpha_2(u)\beta_2(u)$$

Thus,

$$\begin{aligned}
\frac{d}{du}(\boldsymbol{\alpha}(u)\cdot\boldsymbol{\beta}(u)) = \frac{d}{du}(\alpha_1(u)\beta_1(u) + \alpha_2(u)\beta_2(u)) &= \alpha_1'(u)\beta_1(u) + \alpha_1(u)\beta_1'(u) + \alpha_2'(u)\beta_2(u) + \alpha_2(u)\beta_2'(u) = \\
&= (\alpha_1'(u)\beta_1(u) + \alpha_2'(u)\beta_2(u)) + (\alpha_1(u)\beta_1'(u) + \alpha_2(u)\beta_2'(u)) = \boldsymbol{\alpha}'(u)\cdot\boldsymbol{\beta}(u) + \boldsymbol{\alpha}(u)\cdot\boldsymbol{\beta}'(u)
\end{aligned}$$

(b) Since the point $\boldsymbol{\alpha}(u_0)$ is the closest to the origin, the derivative of the function $\|\boldsymbol{\alpha}(u)\|^2$ vanishes at point u_0. Using the equality $\|\boldsymbol{\alpha}(u)\|^2 = \boldsymbol{\alpha}(u)\cdot\boldsymbol{\alpha}(u)$ and (a), we obtain

$$0 = \frac{d}{du}\|\boldsymbol{\alpha}(u)\|^2|_{u_0} = \frac{d}{du}\boldsymbol{\alpha}(u)\cdot\boldsymbol{\alpha}(u) = 2\boldsymbol{\alpha}'(u_0)\cdot\boldsymbol{\alpha}(u_0),$$

so $\alpha'(u_0)$ and $\alpha(u_0)$ are orthogonal.

1.5. The second derivative $\boldsymbol{\alpha}''(u)$ of a smooth plane curve $\boldsymbol{\alpha}(u)$ is identically zero. What can be said about $\boldsymbol{\alpha}$?

Solution: Since $\boldsymbol{\alpha}''(u) \equiv 0$, the tangent vector $\boldsymbol{\alpha}'(u)$ is constant, which implies that $\boldsymbol{\alpha}(u)$ is either a constant speed parametrization of a line or just a single point.

1.6. Let $\boldsymbol{\alpha} : (0, \pi) \to \mathbb{R}^2$ be a curve defined by

$$\boldsymbol{\alpha}(u) = (\sin u, \cos u + \log \tan \frac{u}{2})$$

The trace of $\boldsymbol{\alpha}$ is called a *tractrix*.

(a) Sketch $\boldsymbol{\alpha}$.

(b) Show that a tangent vector at $\boldsymbol{\alpha}(u_0)$ can be written as

$$\boldsymbol{\alpha}'(u_0) = (\cos u_0, -\sin u_0 + \frac{1}{\sin u_0})$$

Show that $\alpha(u)$ is smooth, and it is regular everywhere except $u = \pi/2$.

(c) Write down the equation of a tangent line l_{u_0} to the trace of $\boldsymbol{\alpha}$ at $\boldsymbol{\alpha}(u_0)$.

(d) Show that the distance between $\boldsymbol{\alpha}(u_0)$ and the intersection of l_{u_0} with y-axis is constantly equal to 1.

Solution: The equation of a tangent line l_{u_0} to the trace of $\boldsymbol{\alpha}$ at $\boldsymbol{\alpha}(u_0)$ can be written as $r(v) = \boldsymbol{\alpha}(u_0) + v\boldsymbol{\alpha}'(u_0)$, or

$$r(v) = (\sin u_0 + v \cos u_0, \cos u_0 + \log \tan \frac{u_0}{2} - v \sin u_0 + v\frac{1}{\sin u_0})$$

The square of the distance between $r(v)$ and $\boldsymbol{\alpha}(u_0)$ is equal to $v^2\|\boldsymbol{\alpha}'(u_0)\|^2$. The line intersects y-axis at $v = -\tan u_0$, so (the square of) the required distance is equal to

$$\tan^2 u_0 \|(\cos u_0, -\sin u_0 + \frac{1}{\sin u_0})\|^2 = \tan^2 u_0(\cos^2 u_0 + \sin^2 u_0 - 2 + \frac{1}{\sin^2 u_0}) = \tan^2 u_0(\frac{1}{\sin^2 u_0} - 1) = 1$$