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Differential Geometry III, Solutions 10 (Week 10)

Coordinate curves, angles and area

10.1. Let $\boldsymbol{x} : U \to S$ be a local parametrization of a regular surface S, and denote by E, F, G the coefficients of the first fundamental form in this parametrization. Show that the tangent vector $a\,\partial_u\boldsymbol{x} + b\,\partial_v\boldsymbol{x}$ bisects the angle between the coordinate curves if and only if

$$\sqrt{G}(aE + bF) = \sqrt{E}(aF + bG).$$

Further, if

$$\boldsymbol{x}(u, v) = (u, v, u^2 - v^2),$$

find a vector tangential to S which bisects the angle between the coordinate curves at the point $(1, 1, 0) \in S$.

Solution:

The cosine of the angle of the vector $\boldsymbol{w} = a\,\partial_u\boldsymbol{x} + b\,\partial_v\boldsymbol{x}$ with coordinate curve $v = \text{const}$ is equal to

$$\frac{\langle a\,\partial_u\boldsymbol{x} + b\,\partial_v\boldsymbol{x}, \partial_u\boldsymbol{x}\rangle}{\|w\|\|\partial_u\boldsymbol{x}\|} = \frac{aE + bF}{\|w\|\sqrt{E}}$$

Similarly, the cosine of the angle of $\boldsymbol{w}$ with coordinate curve $u = \text{const}$ is equal to

$$\frac{\langle a\,\partial_u\boldsymbol{x} + b\,\partial_v\boldsymbol{x}, \partial_v\boldsymbol{x}\rangle}{\|w\|\|\partial_v\boldsymbol{x}\|} = \frac{aF + bG}{\|w\|\sqrt{G}}$$

The equality of the cosines

$$\frac{aE + bF}{\|w\|\sqrt{E}} = \frac{aF + bG}{\|w\|\sqrt{G}}$$

is equivalent to

$$\sqrt{G}(aE + bF) = \sqrt{E}(aF + bG)$$

as required.

For

$$\boldsymbol{x}(u, v) = (u, v, u^2 - v^2),$$

we have

$$\begin{aligned} \partial_u\boldsymbol{x}(u, v) &= (1, 0, 2u), \\ \partial_v\boldsymbol{x}(u, v) &= (0, 1, -2v), \end{aligned}$$

which implies that

$$E(u, v) = 1 + 4u^2, \qquad F(u, v) = -4uv, \qquad G(u, v) = 1 + 4v^2.$$

The point $(1, 1, 0)$ has coordinates $(u, v) = (1, 1)$, so we have $E = G = 5$, $F = -4$. Thus, we obtain the following equation on (a, b):

$$\sqrt{5}(5a - 4b) = \sqrt{5}(-4a + 5b),$$

which is equivalent to $a = b$. Thus, the vector $\partial_u\boldsymbol{x} + \partial_v\boldsymbol{x}$ bisects the angle.

10.2. Find two families of curves on the helicoid parametrized by

$$\boldsymbol{x}(u,v) = (v\cos u, v\sin u, u)$$

which, at each point, bisect the angles between the coordinate curves.
(Show that they are given by $u \pm \sinh^{-1} v = c$, where c is a constant on each curve in the family.)

Solution: We have

$$\begin{aligned} \partial_u \boldsymbol{x}(u,v) &= (-v\sin u, v\cos u, 1), \\ \partial_v \boldsymbol{x}(u,v) &= (\cos u, \sin u, 0), \end{aligned}$$

which implies that

$$E(u,v) = 1 + v^2, \qquad F(u,v) = 0, \qquad G(u,v) = 1,$$

so the equation from Exercise 10.1 becomes

$$a\sqrt{v^2+1} = b.$$

The curve $u - \sinh^{-1} v = c$ can be parametrized by $\boldsymbol{\alpha}(u) = (u, \sinh(u-c))$, so

$$\boldsymbol{\alpha}'(u,v) = \partial_u \boldsymbol{x} + \cosh(u-c)\partial_v \boldsymbol{x} = \partial_u \boldsymbol{x} + \cosh(u-c)\partial_v \boldsymbol{x} = \partial_u \boldsymbol{x} + \sqrt{v^2+1}\,\partial_v \boldsymbol{x}$$

as required.
The curve $u + \sinh^{-1} v = c$ can be parametrized by $\boldsymbol{\beta}(u) = (u, -\sinh(u-c))$, so

$$\boldsymbol{\beta}'(u,v) = \partial_u \boldsymbol{x} - \cosh(u-c)\partial_v \boldsymbol{x} = \partial_u \boldsymbol{x} - \cosh(u-c)\partial_v \boldsymbol{x} = \partial_u \boldsymbol{x} - \sqrt{v^2+1}\,\partial_v \boldsymbol{x}.$$

Then

$$\langle \boldsymbol{\alpha}', \boldsymbol{\beta}' \rangle = E - (v^2+1)G = 0,$$

which implies that $\boldsymbol{\beta}'$ bisects the angle between $\partial_u \boldsymbol{x}$ and $-\partial_v \boldsymbol{x}$.

10.3. The coordinate curves of a parametrization $\boldsymbol{x}(u,v)$ constitute a *Chebyshev net* if the lengths of the opposite sides of any quadrilateral formed by them are equal.
(a) Show that a necessary and sufficient condition for this is

$$\frac{\partial E}{\partial v} = \frac{\partial G}{\partial u} = 0.$$

(b) Show that if coordinate curves constitute a Chebyshev net, then it is possible to reparametrize the coordinate neighborhood in such a way that the new coefficients of the first fundamental form are

$$E = 1, \qquad F = \cos\vartheta, \qquad G = 1,$$

where ϑ is the angle between coordinate curves.

Solution:
(a) Assume that coordinate curves constitute a Chebyshev net. Consider a quadrilateral with vertices $(u_0,v_0), (u_1,v_0), (u_0,v_1), (u_1,v_1)$ formed by coordinate curves. The length of the side with vertices $(u_0,v_1), (u_1,v_1)$ is equal to

$$\int\limits_{u_0}^{u_1} \|\partial_u \boldsymbol{x}(u,v_1)\|\,\mathrm{d}u = \int\limits_{u_0}^{u_1} \sqrt{E(u,v_1)}\,\mathrm{d}u$$

Thus, the integral $\int_{u_0}^{u_1} \sqrt{E(u,v_1)}\,\mathrm{d}u$ does not depend on v_1, i.e. it is a function of u_1 only. Differentiating it by u_1, we see that $\sqrt{E(u_1,v_1)}$ is also a function of u_1 only, so $E(u,v)$ does not depend on v. The considerations for G are similar, and the converse statement is straightforward.
(b) Take

$$\tilde{u}(u) = \int \sqrt{E(u)}\,\mathrm{d}u$$

Then $\tilde{u}$ is parametrized by arc length, so $\tilde{E}(\tilde{u}) \equiv 1$. Similarly, we can make $\tilde{G}(\tilde{v}) \equiv 1$. Now F is equal to the cosine of the angle by definition.

10.4. Show that a surface of revolution can always be parametrized so that

$$E = E(v), \qquad F = 0, \qquad G = 1$$

Solution: Parametrize the surface by

$$\boldsymbol{x} = (f(v)\cos u, f(v)\sin u, g(v)),$$

where $\boldsymbol{\alpha}(v) = (f(v), 0, g(v))$ is the generating curve. Then

$$\begin{aligned} \partial_u \boldsymbol{x} &= (-f(v)\sin u, f(v)v\cos u, 0), \\ \partial_v \boldsymbol{x} &= (f'(v)\cos u, f'(v)\sin u, g'(v)), \end{aligned}$$

which implies that

$$E(u,v) = f^2(v), \qquad F(u,v) = 0, \qquad G(u,v) = f'^2(v) + g'^2(v) = \|\boldsymbol{\alpha}'\|^2$$

Parametrizing $\boldsymbol{\alpha}(v)$ by arc length we obtain a required parametrization of the surface.

10.5. Let S be the surface $\{(x,y,z) \in \mathbb{R}^3 \,|\, z = x^2 - y^2\}$ and let $\mathcal{F}$ be the family of curves on S obtained as the intersection of S with the planes $z =$ const. Find the family of curves on S which meet $\mathcal{F}$ orthogonally and show that they are the intersections of S with the family of hyperbolic cylinders $xy =$ const.

Solution:

A (part of a) curve $x^2 - y^2 = c_1$ on S can be parametrized by $\boldsymbol{\alpha}(y) = (\sqrt{y^2+c}, y)$, so

$$\boldsymbol{\alpha}'(y) = \frac{y}{\sqrt{y^2+c_1}}\partial_x + \partial_y = \frac{y}{x}\partial_x + \partial_y$$

A curve $xy = c_2$ on S can be paranetrized by $\boldsymbol{\beta}(x) = (x, \frac{c}{x})$, so

$$\boldsymbol{\beta}'(x) = \partial_x - \frac{c_2}{x^2}\partial_y = \partial_x - \frac{y}{x}\partial_y$$

Now we recall that the coefficients of the first fundamental form found in Exercise 10.1 are

$$E(u,v) = 1 + 4x^2, \qquad F(u,v) = -4xy, \qquad G(u,v) = 1 + 4y^2,$$

so we compute the inner product of $\boldsymbol{\alpha}'$ and $\boldsymbol{\beta}'$ to get

$$\begin{aligned} \langle \boldsymbol{\alpha}', \boldsymbol{\beta}' \rangle = \left\langle \frac{y}{x}\partial_x + \partial_y, \partial_x - \frac{y}{x}\partial_y \right\rangle = \frac{y}{x}E + F - \frac{y^2}{x^2}F - \frac{y}{x}G = \\ = \frac{y}{x}(1+4x^2) + 4xy\left(\frac{y^2}{x^2} - 1\right) - \frac{y}{x}(1+4y^2) = \frac{y}{x} + 4xy + \frac{4y}{x} - 4xy - \frac{y}{x} - \frac{4y}{x} = 0 \end{aligned}$$

Note that we could avoid computations on S: one could consider $\boldsymbol{\alpha}$ and $\boldsymbol{\beta}$ as curves in $\mathbb{R}^3$, and keeping in mind that z-coordinate of $\boldsymbol{\alpha}'$ is equal to zero, the dot product of $\boldsymbol{\alpha}'$ and $\boldsymbol{\beta}'$ is equal to $\langle (\frac{y}{x}, 1, 0) \cdot (1, -\frac{y}{x}, z'(x)) \rangle = 0$.

10.6. Using the notation of Exercise 10.2, show that the family of curves orthogonal to the family

$$v \cos u = \text{const}$$

is the family defined by $(1+v^2)\sin^2 u = \text{const}$.

Solution:

The coefficients of the first fundamental form found in Exercise 10.2 are

$$E(u,v) = 1 + v^2, \qquad F(u,v) = 0, \qquad G(u,v) = 1.$$

A curve $v \cos u = c_1$ on S can be parametrized by $\boldsymbol{\alpha}(u) = (u, c_1 / \cos u)$, so

$$\boldsymbol{\alpha}'(u) = (1, -c_1 \sin u / \cos^2 u) = (1, -v \tan u).$$

A curve $(1 + v^2) \sin^2 u = c_2$ on S can be paranetrized by $\boldsymbol{\beta}(u) = \left(u, -\sqrt{\dfrac{c_2}{\sin^2 u} - 1}\right)$, so

$$\boldsymbol{\beta}'(u) = \left(1, \frac{1}{\tan u}\left(v + \frac{1}{v}\right)\right).$$

Computing the inner product of $\boldsymbol{\alpha}'$ and $\boldsymbol{\beta}'$ we obtain

$$\langle \boldsymbol{\alpha}', \boldsymbol{\beta}' \rangle = E - v \tan u \frac{1}{\tan u}\left(v + \frac{1}{v}\right) = 1 + v^2 - (v^2 + 1) = 0.$$