Durham University
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Epiphany 2017

Differential Geometry III, Homework 1 (Week 11)

Due date for starred problems: **Thursday, February 2**.

Isometries and conformal maps - 1

1.1. Let $a > 0$. Construct explicitly a local isometry from the plane $P = \{ (u, v, 0) \in \mathbb{R}^3 \mid u, v \in \mathbb{R} \}$ onto the cylinder $S = \{ (x, y, z) \in \mathbb{R}^3 \mid x^2 + y^2 = a^2 \}$.

1.2. ($\star$) Let b be a positive number such that $\sqrt{1 + b^2}$ is an integer n. Let S be the circular cone obtained by rotating the curve given by $\boldsymbol{\alpha}(v) = (v, 0, bv)$, $v > 0$, about the z-axis. Let the coordinate xy-plane P be parametrized by polar coordinates (r, ϑ):

$$\boldsymbol{x}\colon U = (0, \infty) \times (0, 2\pi) \longrightarrow P, \quad \boldsymbol{x}(r, \vartheta) = (r \cos \vartheta, r \sin \vartheta, 0).$$

Show that the map $f\colon P \setminus \{(0, 0, 0)\} \longrightarrow S$ defined on $\boldsymbol{x}(U)$ by

$$f(\boldsymbol{x}(r, \vartheta)) = \frac{1}{n}\big(r \cos n\vartheta, r \sin n\vartheta, br\big)$$

is a local isometry on $\boldsymbol{x}(U)$.

1.3. Let S_1, S_2, S_3 be regular surfaces.

(a) Suppose that $f\colon S_1 \longrightarrow S_2$ and $g\colon S_2 \longrightarrow S_3$ are local isometries. Prove that $g \circ f\colon S_1 \longrightarrow S_3$ is a local isometry.

(b) Suppose that $f\colon S_1 \longrightarrow S_2$ and $g\colon S_2 \longrightarrow S_3$ are conformal maps with conformal factors $\lambda\colon S_1 \longrightarrow (0, \infty)$ and $\mu\colon S_2 \longrightarrow (0, \infty)$, respectively. Prove that $g \circ f\colon S_1 \longrightarrow S_3$ is a conformal map and calculate its conformal factor. (The conformal factor of a conformal map is the function appearing as factor in front of the inner product in the definition.)

(c) Let f and g be conformal maps with conformal factors λ and μ as in the previous part. Find a condition on λ and μ such that $g \circ f$ is a *(local) isometry.*

1.4. Let S be the surface of revolution parametrized by

$$\boldsymbol{x}(u, v) = \Big(\cos v \cos u, \cos v \sin u, -\sin v + \log \tan\Big(\frac{\pi}{4} + \frac{v}{2}\Big)\Big),$$

where $0 < u < 2\pi, 0 < v < \pi/2$. Let S_1 be the region

$$S_1 = \{ \boldsymbol{x}(u, v) \mid 0 < u < \pi, 0 < v < \pi/2 \}$$

and let S_2 be the region

$$S_2 = \{ \boldsymbol{x}(u, v) \mid 0 < u < 2\pi, \pi/3 < v < \pi/2 \}.$$

Show that the map

$$\boldsymbol{x}(u, v) \mapsto \boldsymbol{x}\Big(2u, \arccos\Big(\frac{1}{2} \cos v\Big)\Big)$$

is an isometry from S_1 onto S_2.