Durham University
Pavel Tumarkin

Epiphany 2017

Differential Geometry III, Solutions 3 (Week 13)

Weingarten map, Gauss, mean and principal curvatures - 1

3.1. A local parametrization $\boldsymbol{x}$ of a surface S in $\mathbb{R}^3$ is called *orthogonal* provided $F = 0$ (so $\boldsymbol{x}_u$ and $\boldsymbol{x}_v$ are orthogonal at each point). It is called *principal* if $F = 0$ and $M = 0$, where E, F, G (resp. L, M, N) are the coefficients of the first (resp. second) fundamental form.

(a) Let $\boldsymbol{x}$ be an *orthogonal* parametrization. Show that, at any point $p = \boldsymbol{x}(u, v)$ on S,

$$-d\boldsymbol{N}_p(\boldsymbol{x}_u) = \frac{L}{E}\boldsymbol{x}_u + \frac{M}{G}\boldsymbol{x}_v, \qquad -d\boldsymbol{N}_p(\boldsymbol{x}_v) = \frac{M}{E}\boldsymbol{x}_u + \frac{N}{G}\boldsymbol{x}_v,$$

where $\boldsymbol{N}$ denotes the Gauss map.

(b) Assume now that the parametrization is *principal*. Show that $\kappa_1 = L/E$ and $\kappa_2 = N/G$ are the principal curvatures. Calculate the Gauss and mean curvature in terms of E, G, L, N. Determine the principal directions.

Solution:

(a) Since $d_p\boldsymbol{N}$ maps T_pS into T_pS, we can express $-d_p\boldsymbol{N}(\boldsymbol{x}_u)$ and $-d_p\boldsymbol{N}(\boldsymbol{x}_v)$ as a linear combination of $\boldsymbol{x}_u$ and $\boldsymbol{x}_v$, i.e.,

$$-d_p\boldsymbol{N}(\boldsymbol{x}_u) = a\boldsymbol{x}_u + b\boldsymbol{x}_v \quad \text{and} \quad -d_p\boldsymbol{N}(\boldsymbol{x}_v) = c\boldsymbol{x}_u + d\boldsymbol{x}_v.$$

Multiplying both equations with $\cdot\boldsymbol{x}_u$ and $\cdot\boldsymbol{x}_v$ gives (using the definitions of the coefficients of the first and second fundamental forms and the equalities $\boldsymbol{N}_u \cdot \boldsymbol{x}_u + \boldsymbol{N} \cdot \boldsymbol{x}_{uu} = 0$ etc.)

$$L = aE + bF, \quad M = aF + bG, \quad M = cE + dF, \quad N = cF + dG,$$

and, since $F = 0$,

$$a = \frac{L}{E}, \quad b = \frac{M}{G}, \quad c = \frac{M}{E}, \quad d = \frac{N}{G},$$

i.e., the desired equation.

(b) If $M = 0$, then the equations from the first part are

$$-d\boldsymbol{N}_p(\boldsymbol{x}_u) = \frac{L}{E}\boldsymbol{x}_u \quad \text{and} \quad -d\boldsymbol{N}_p(\boldsymbol{x}_v) = \frac{N}{G}\boldsymbol{x}_v.$$

Therefore, $\boldsymbol{x}_u$ is an eigenvector with eigenvalue L/E, as well as $\boldsymbol{x}_v$ with eigenvalue N/G. Hence the principal, Gauss and mean curvatures are

$$\kappa_1 = \frac{L}{E}, \quad \kappa_2 = \frac{N}{G}, \quad K = \kappa_1\kappa_2 = \frac{LN}{EG}, \quad H = \frac{1}{2}(\kappa_1 + \kappa_2) = \frac{L}{2E} + \frac{N}{2G} = \frac{LG + NE}{2EG}.$$

3.2. Calculation of the Weingarten map directly for surfaces of revolution

Let $f: J \longrightarrow (0, \infty)$ and $g: J \longrightarrow \mathbb{R}$ be smooth functions on some open interval J in $\mathbb{R}$ and let $\boldsymbol{\alpha}: J \longrightarrow \mathbb{R}^3$ be a space curve given by $\boldsymbol{\alpha}(v) = (f(v), 0, g(v))$. Assume that this curve is parametrized by arc length. Let S be the surface of revolution obtained by rotating $\boldsymbol{\alpha}$ around the z-axis.

(a) Find suitable parametrizations $\boldsymbol{x}: U_i \longrightarrow S$ of S and determine parameter domains U_1 and U_2 covering the whole surface S. Calculate the normal vector $\boldsymbol{N}$ at $\boldsymbol{x}(u, v)$

(b) Express $a, b, c, d \in \mathbb{R}$ in $-d\boldsymbol{N}_p(\boldsymbol{x}_u) = a\boldsymbol{x}_u + b\boldsymbol{x}_v$ and $-d\boldsymbol{N}_p(\boldsymbol{x}_v) = c\boldsymbol{x}_u + d\boldsymbol{x}_v$ in terms of f and g.

(c) Calculate the principal directions and principal curvatures.

(d) Calculate the Gauss and mean curvatures.

(e) Compare your results with Example 9.13 from the lectures.

Solution: The generating curve is parametrized by arc length, so $(f')^2 + (g')^=1$.

(a) The standard parametrization of a surface of revolution is given by

$$\boldsymbol{x}(u, v) = (f(v)\cos u, f(v)\sin u, g(v)), \qquad (u, v) \in U$$

where $U = U_1$ or $U = U_2$ and (for example)

$$U_1 = (-\pi, \pi) \times J, \qquad U_2 = (0, 2\pi) \times J,$$

so that the first (angular) variable u covers all angles.
Make sure you understand why we need (at least) two parameter sets U_1 and U_2.
Moreover, (f, g have the argument v, and cos, sin have the argument u)

$$\boldsymbol{x}_u = (-f\sin, f\cos, 0), \qquad \boldsymbol{x}_v = (f'\cos, f'\sin, g'),$$

hence $\boldsymbol{x}_u \times \boldsymbol{x}_v = (g'\cos, g'\sin, -f')$. Since the generating curve is parametrized by arc length, $\boldsymbol{x}_u \times \boldsymbol{x}_v$ is a unit vector, so

$$\boldsymbol{N} = (g'\cos, g'\sin, -f').$$

Moreover,

$$E = \boldsymbol{x}_u \cdot \boldsymbol{x}_u = f^2, \qquad F = 0, \qquad G = (f')^2 + (g')^2 = 1.$$

We also need (later on) the coefficients of the second fundamental form, so we calculate

$$\boldsymbol{x}_{uu} = (-f\cos, -f\sin, 0), \qquad \boldsymbol{x}_{uv} = (-f'\sin, f'\cos, 0), \qquad \boldsymbol{x}_{vv} = (f''\cos, f''\sin, g'')$$

so that

$$L = \boldsymbol{x}_{uu} \cdot \boldsymbol{N} = -fg', \qquad M = \boldsymbol{x}_{uv} \cdot \boldsymbol{N} = 0, \qquad N = \boldsymbol{x}_{vv} \cdot \boldsymbol{N} = f''g' - f'g''$$

(b) We multiply both equations with $\boldsymbol{x}_u$ and $\boldsymbol{x}_v$, so that

$$\begin{aligned} L &= -d_p\boldsymbol{N}(\boldsymbol{x}_u) \cdot \boldsymbol{x}_u = aE + bF, & M &= -d_p\boldsymbol{N}(\boldsymbol{x}_u) \cdot \boldsymbol{x}_v = aF + bG, \\ M &= -d_p\boldsymbol{N}(\boldsymbol{x}_v) \cdot \boldsymbol{x}_u = cE + dF, & N &= -d_p\boldsymbol{N}(\boldsymbol{x}_v) \cdot \boldsymbol{x}_v = cF + dG, \end{aligned}$$

where we used the equalities $\boldsymbol{N}_u \cdot \boldsymbol{x}_u + \boldsymbol{N} \cdot \boldsymbol{x}_{uu} = 0$ etc.
The above equations simplify to

$$\begin{aligned} L &= aE, & M &= bG, \\ M &= cE, & N &= dG. \end{aligned}$$

If $F = 0$, then

$$a = \frac{L}{E}, \qquad b = \frac{M}{G}, \qquad c = \frac{M}{E}, \qquad d = \frac{N}{G}.$$

If, in addition, $M = 0$, then

$$a = \frac{L}{E}, \qquad b = 0, \qquad c = 0, \qquad d = \frac{N}{G}.$$

(c) We have (using the above expressions for a, b, c and d)

$$-d_p\boldsymbol{N}(\boldsymbol{x}_u) = \frac{L}{E}\boldsymbol{x}_u \qquad \text{and} \qquad -d_p\boldsymbol{N}(\boldsymbol{x}_v) = \frac{N}{G}\boldsymbol{x}_v,$$

hence the basis vectors $\boldsymbol{x}_u$ and $\boldsymbol{x}_v$ are eigenvectors (principal directions) with eigenvalues (principal curvatures)

$$\kappa_1 = \frac{L}{E} = \frac{-fg'}{f^2} = -\frac{g'}{f} \qquad \text{and} \qquad \kappa_2 = \frac{N}{G} = f''g' - f'g''$$

(d) The Gauss and mean curvature are

$$K = \kappa_1 \kappa_2 = \frac{g'(f'g'' - f''g')}{f} \quad \text{and} \quad H = \frac{1}{2}(\kappa_1 + \kappa_2) = -\frac{g'}{2f} + \frac{1}{2}(f''g' - f'g'').$$

3.3. Let S be the surface in $\mathbb{R}^3$ defined by the equation

$$z = \frac{1}{1 + x^2 + y^2}.$$

Find the principal curvatures and the umbilic points (i.e., the points where the principal curvatures are the same). Give a sketch of the surface showing the regions of the surface where the Gauss curvature K is strictly positive and strictly negative.

Solution:

Consider S as a surface of revolution with the standard parametrization given by $\boldsymbol{x}(u, v) = (f(v)\cos u, f(v)\sin u, g(v))$ with functions f and g to be determined. That $\boldsymbol{x}(u, v)$ is an element of the surface $S = \{\, (x, y, z) \,|\, z = 1/(1 + x^2 + y^2) \,\}$ means that

$$g(v) = \frac{1}{1 + f(v)^2}.$$

Choose e.g. $f(v) = v$ then $g(v) = 1/(1 + v^2)$. As a parameter domain U we choose $U_1 = (-\pi, \pi) \times (0, \infty)$ and $U_2 = (0, 2\pi) \times (0, \infty)$.

Note: This parametrization covers all points on S *except* the point $(0, 0, 1) \in S$.

Calculating the coefficients of the first and second fundamental forms, we obtain

$$E = f^2 = v^2, \qquad F = 0, \qquad G = f'^2 + g'^2 = 1 + \frac{4}{v}^2 (1 + v^2)^2$$

$$L = \frac{-fg'}{\sqrt{f'^2 + g'^2}}, \qquad M = 0, \qquad N = \frac{f''g' - f'g''}{\sqrt{f'^2 + g'^2}}$$

(see Example 9.13). In our concrete case, we have

$$f'(v) = 1, \quad f''(v) = 0, \quad g'(v) = \frac{-2v}{(1 + v^2)^2}, \quad g''(v) = \frac{-2(1 + v^2) + 2v(2v)2}{(1 + v^2)^3} = \frac{2(3v^2 - 1)}{(1 + v^2)^3}.$$

Since the parametrization is *principal* ($F = 0$ and $M = 0$), the principal curvatures are

$$\kappa_1 = \frac{L}{E} = \frac{-fg'}{f^2((f')^2 + (g')^2)^{1/2}} = -\frac{g'}{f((f')^2 + (g')^2)^{1/2}}, \qquad \kappa_2 = \frac{N}{G} = \frac{(f''g' - f'g'')}{((f')^2 + (g')^2)^{3/2}},$$

which means here that

$$\kappa_1 = \frac{2}{(1 + v^2)^2 \left(1 + \frac{4v^2}{(1+v^2)^4}\right)^{1/2}} \quad \text{and} \quad \kappa_2 = -\frac{2(3v^2 - 1)}{(1 + v^2)^3 \left(1 + \frac{4v^2}{(1+v^2)^4}\right)^{3/2}}.$$

Now, a point is umbilic if $\kappa_1 = \kappa_2$ at this point, i.e., if

$$1 = -\frac{(3v^2 - 1)}{(1 + v^2)\left(1 + \frac{4v^2}{(1+v^2)^4}\right)},$$

or, equivalently, $(v > 0)$

$$\begin{aligned} 0 &= (1 + v^2)\left(1 + \frac{4v^2}{(1 + v^2)^4}\right) + (3v^2 - 1) \\ &= 4v^2 + \frac{4v^2}{(1 + v^2)^3} \end{aligned}$$

which has no solution if $v \neq 0$. Therefore, the surface has no umbilic point *on the points covered by the parametrization as surface of revolution*, i.e., the points $p \in S \setminus \{(0, 0, 1)\}$ are not umbilic.

What about the point $(0,0,1)$*?*

If we are just at the point $(0,0,1)$ (with parameter values $(u,v)=(0,0)$ in the parametrization given by $\boldsymbol{x}(u,v)=(u,v,1/(1+u^2+v^2)))$, we obtain

$$f(x,y)=\frac{1}{1+x^2+y^2},\qquad f_x(x,y)=\frac{-2x}{(1+x^2+y^2)^2},\qquad f_y(x,y)=\frac{-2y}{(1+x^2+y^2)^2},$$

and

$$f_{xx}(x,y)=\frac{-2(1+x^2+y^2)+2x2x2}{(1+x^2+y^2)^3}=\frac{-2(1-3x^2+y^2)}{(1+x^2+y^2)^3}$$

and similarly

$$f_{xy}(x,y)=\frac{(-2)(-2x)(2y)}{(1+x^2+y^2)^3}=\frac{8xy}{(1+x^2+y^2)^3},\qquad f_{yy}(x,y)=\frac{-2(1+x^2-3y^2)}{(1+x^2+y^2)^3}.$$

Hence, we obtain for the coefficients of the first and second fundamental form at $(0,0)$ the expressions

$$E(0,0)=1+f_x(0,0)=1,\qquad F(0,0)=f_x(0,0)f_y(0,0)=0,\qquad G(0,0)=1+f_y(0,0)=1.$$

Denote $D=1+f_x^2(0,0)+f_y^2(0,0)=1$, then

$$L(0,0)=\frac{f_{xx}(0,0)}{D}=-2,\qquad M(0,0)=\frac{f_{xy}(0,0)}{D}=0,\qquad N(0,0)=\frac{f_{yy}(0,0)}{D}=-2.$$

Therefore, the Gauss and mean curvatures at the parameter value $(0,0)$ are

$$K=\frac{LN-M^2}{EG-F^2}=4,\qquad H=\frac{EN-2FM+GL}{2(EG-F^2)}=\frac{-2-2}{2}=-2,$$

so that the principal curvatures are the roots of

$$\kappa^2-2H\kappa+K=0,\qquad\text{or}\qquad \kappa^2+4+4=(\kappa+2)^2=0,$$

i.e., $\kappa_1=\kappa_2=-2$.

Therefore, $(0,0,1)$ is the only umbilic point of the surface (as one might already guess from the rotational symmetry of the surface).

One could start with this parametrization (as a graph) right from the beginning, but it seems that the formulas for the two principal curvaturs become much more complicated than as for a surface of revolution.

3.4. ($\star$) The pseudosphere

The pseudosphere is the surface of revolution obtained by rotating the tractrix with parametrization $\boldsymbol{\alpha}(s)=(1/\cosh s,0,s-\tanh s)$ around the z-axis. Prove that the pseudosphere has constant Gauss curvature $K=-1$.

Solution:

Calculating the coefficients of the first and second fundamental forms, we obtain

$$E=f^2,\qquad F=0,\qquad G=f'^2+g'^2$$

$$L=\frac{-fg'}{\sqrt{f'^2+g'^2}},\qquad M=0,\qquad N=\frac{f''g'-f'g''}{\sqrt{f'^2+g'^2}}$$

(see Example 9.13). Let us assume that $v>0$ (the surface for negative values v is just the mirror image w.r.t. the xy-plane).

In our case, we have

$$f(v)=\frac{1}{\cosh v},\qquad f'(v)=-\frac{\sinh v}{\cosh^2 v},\qquad f''(v)=-\frac{\cosh^2 v-2\sinh^2 v}{\cosh^3 v}=\frac{\cosh^2 v-2}{\cosh^3 v},$$

and

$$g(v) = v - \tanh v, \qquad g'(v) = 1 - \frac{1}{\cosh^2 v} = \frac{\cosh^2 v - 1}{\cosh^2 v} = \frac{\sinh^2 v}{\cosh^2 v}, \qquad g''(v) = \frac{2\sinh v}{\cosh^3 v}$$

Moreover, we have

$$f'(v)^2 + g'(v)^2 = \frac{\sinh^2 v + \sinh^4 v}{\cosh^4 v} = \frac{\sinh^2 v(1+\sinh^2 v)}{\cosh^4 v} = \frac{\sinh^2 v \cosh^2 v}{\cosh^4 v} = \frac{\sinh^2 v}{\cosh^2 v} = \tanh^2 v$$

so that

$$\begin{aligned} E &= \frac{1}{\cosh^2 v}, & F &= 0, & G &= \tanh^2 v \\ L &= \frac{-\tanh^2 v/\cosh v}{\tanh v}, & M &= 0, & N &= \frac{f''g' - f'g''}{\sqrt{f'^2 + g'^2}} \\ &= -\frac{\sinh v}{\cosh^2 v}, & & & &= \frac{(\cosh^2 v - 2)\tanh^2 v/\cosh^3 v + 2\sinh^2 v/\cosh^5 v}{\tanh v} \\ & & & & &= \frac{\sinh v}{\cosh^2 v} \end{aligned}$$

Since the parametrization is *principal* ($F = 0$ and $M = 0$), the principal curvatures are

$$\begin{aligned} \kappa_1 &= \frac{L}{E} = -\frac{\sinh v}{\cosh^2 v \cosh^{-2} v} = -\sinh v, \\ \kappa_2 &= \frac{N}{G} = \frac{\sinh v}{\cosh^2 v \tanh^2 v} = \frac{1}{\sinh v}, \end{aligned}$$

hence the Gauss curvature is $K = \kappa_1\kappa_2 = -1$, as desired.