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Differential Geometry III, Solutions 5 (Week 15)

Christoffel symbols and Gauss' Theorema Egregium

5.1. Show that the Gauss curvature K of the surface of revolution locally parametrized by

$$\boldsymbol{x}(u,v) = (f(v)\cos(u), f(v)\sin(u), g(v)), \qquad (u,v) \in U,$$

(for some suitable parameter domain U) is given by

$$K = \frac{1}{2ff'}\Big(\frac{1}{1+(f'/g')^2}\Big)'.$$

If the generating curve is parametrized by arc length, show that $K = -f''/f$. Deduce Theorema Egregium in the latter case.

Solution: We have already calculated the coefficients of the first and second fundamental forms for a surface of revolution (see e.g. Example 9.13), so we just cite the result here again:

$$E = f^2, \qquad F = 0, \qquad G = f'^2 + g'^2$$

$$L = \frac{-fg'}{\sqrt{f'^2+g'^2}}, \qquad M = 0, \qquad N = \frac{f''g' - f'g''}{\sqrt{f'^2+g'^2}}.$$

If we assume now that $f(v) > 0$ everywhere, then we have

$$\kappa_1 = \frac{L}{E} = \frac{-g'}{f(f'^2+g'^2)^{1/2}}, \qquad \kappa_2 = \frac{N}{G} = \frac{f''g' - f'g''}{(f'^2+g'^2)^{3/2}}$$

(see Prop. 9.12), hence the Gauss curvature is

$$K = \kappa_1\kappa_2 = \frac{-g'(f''g'-f'g'')}{f(f'^2+g'^2)^2} = \frac{-(f'/g')'}{fg'\big((f'/g')^2+1\big)^2}$$
$$= \frac{-2(f'/g')(f'/g')'}{2f'f\big((f'/g')^2+1\big)^2} = \frac{1}{2ff'}\Big(\frac{1}{(f'/g')^2+1}\Big)'$$

as desired (we have also implicitly assumed here that $f'(v) \neq 0 \neq g'(v)$). If the curve is parametrized by arc length, then $f'^2 + g'^2 = 1$, and

$$\kappa_1 = \frac{L}{E} = \frac{-g'}{f}, \qquad \kappa_2 = \frac{N}{G} = f''g' - f'g''.$$

Moreover, differentiating $f'^2 + g'^2 = 1$ gives $f'f'' + g'g'' = 0$, and we obtain

$$K = \kappa_1\kappa_2 = -\frac{g'(f''g'-f'g'')}{f} = -\frac{g'f''g' + f'(f'f'')}{f} = -\frac{(g'^2+f'^2)f''}{f} = -\frac{f''}{f}$$

as desired (this could also be obtained by simplifying the formula for K obtained above).

Now we can deduce Gauss' Theorema Egregium by expressing f in terms of the coefficients of the first fundamental form: $f = \sqrt{E}$. Then we have

$$f' = \frac{E_v}{2\sqrt{E}}, \qquad f'' = \frac{E_{vv}E - E_v^2/2}{2E^{3/2}},$$

so that

$$-f''/f = \frac{E_v^2}{4E^2} - \frac{E_v v}{2E} = -\frac{1}{2\sqrt{E}}\Big(\frac{E_v}{\sqrt{E}}\Big)_v$$

which is a special case ($G = 1$, i.e, $G_u = 0$) of the formula of Example 10.9.

5.2. Let $\boldsymbol{x}\colon U \longrightarrow S$ be a parametrization of a surface S for which $E = G = 1$ and $F = \cos(uv)$ (so that uv is the angle between the coordinate curves). Determine a suitable parameter domain U on which $\boldsymbol{x}(U)$ is a surface (i.e., where the coordinate curves are not tangential). Show that

$$K = -\frac{1}{\sin(uv)}.$$

Solution:

Suitable parameter domain: The tangent vectors $\boldsymbol{x}_u$ and $\boldsymbol{x}_v$ are linearly dependent iff $F = \cos\vartheta = \pm 1$, where $\vartheta = uv$.

Another way to see this restriction is as follows: we have to assure that $\begin{pmatrix} 1 & \cos\vartheta \\ \cos\vartheta & 1 \end{pmatrix}$ with $\vartheta = uv$ is a positive definite matrix. Its determinant is $1 - \cos^2\vartheta$, and this is positive iff $\cos\vartheta \neq \pm 1$. Since its trace is always positive (the trace is 2), the matrix is positive definite iff $\cos\vartheta \neq \pm 1$.

So a maximal parameter domain could be

$$U := \{\, (u, v) \in \mathbb{R}^2 \mid uv \notin \pi\mathbb{Z} \,\},$$

or, if you prefer a connected domain, another choice could be

$$U := \{\, (u, v) \in \mathbb{R}^2 \mid 0 < u < \pi/v, \quad v > 0 \,\}.$$

(choosing just the component $0 < uv < \pi$).

The further calculations are similar to ones used in Example 10.7 (and in the proof of Theorema Egregium). We amend the order a bit to avoid computations with some zeros, and thus to save time in this way.

(a) *Step 1: Christoffel symbols* Γ_{ij}^k are functions defined by

$$\boldsymbol{x}_{uu} = \Gamma_{11}^1 \boldsymbol{x}_u + \Gamma_{11}^2 \boldsymbol{x}_v + L\boldsymbol{N} \qquad (\Gamma 1)$$
$$\boldsymbol{x}_{uv} = \Gamma_{12}^1 \boldsymbol{x}_u + \Gamma_{12}^2 \boldsymbol{x}_v + M\boldsymbol{N} \qquad (\Gamma 2)$$
$$\boldsymbol{x}_{vv} = \Gamma_{22}^1 \boldsymbol{x}_u + \Gamma_{22}^2 \boldsymbol{x}_v + N\boldsymbol{N} \qquad (\Gamma 3)$$

(and we have $\Gamma_{12}^k = \Gamma_{21}^k$ since $\boldsymbol{x}_{uv} = \boldsymbol{x}_{vu}$).
Before calculating Γ_{ij}^k in terms of E, F and G, let us first see what we need (to save some time).
But we also need the following: Express $\boldsymbol{x}_{uu} \cdot \boldsymbol{x}_u$ etc. in terms of E, F, G:

$$\boldsymbol{x}_{uu} \cdot \boldsymbol{x}_u = \frac{1}{2}(\boldsymbol{x}_u \cdot \boldsymbol{x}_u)_u = \frac{1}{2}E_u \quad (= 0) \qquad (1)$$
$$\boldsymbol{x}_{uv} \cdot \boldsymbol{x}_u = \frac{1}{2}(\boldsymbol{x}_u \cdot \boldsymbol{x}_u)_v = \frac{1}{2}E_v \quad (= 0) \qquad (2)$$
$$\boldsymbol{x}_{uv} \cdot \boldsymbol{x}_v = \frac{1}{2}(\boldsymbol{x}_v \cdot \boldsymbol{x}_v)_u = \frac{1}{2}G_u \quad (= 0) \qquad (3)$$
$$\boldsymbol{x}_{vv} \cdot \boldsymbol{x}_v = \frac{1}{2}(\boldsymbol{x}_v \cdot \boldsymbol{x}_v)_v = \frac{1}{2}G_v \quad (= 0) \qquad (4)$$
$$\boldsymbol{x}_{uu} \cdot \boldsymbol{x}_v = (\boldsymbol{x}_u \cdot \boldsymbol{x}_v)_u - \boldsymbol{x}_u \cdot \boldsymbol{x}_{uv} = F_u - \frac{1}{2}E_v \quad (= -v\sin(uv)) \qquad (5)$$
$$\boldsymbol{x}_{vv} \cdot \boldsymbol{x}_u = (\boldsymbol{x}_v \cdot \boldsymbol{x}_u)_v - \boldsymbol{x}_v \cdot \boldsymbol{x}_{uv} = F_v - \frac{1}{2}G_u \quad (= -u\sin(uv)) \qquad (6)$$

(the terms in parentheses correspond to our special case $E = G = 1$, $F = \cos(uv)$).
Multiplying the defining equations for Γ_{ij}^k by $\cdot\boldsymbol{x}_u$ and $\cdot\boldsymbol{x}_v$, we obtain equations

$$E\Gamma_{11}^1 + F\Gamma_{11}^2 = \frac{1}{2}E_u, \qquad E\Gamma_{12}^1 + F\Gamma_{12}^2 = \frac{1}{2}E_v, \qquad E\Gamma_{22}^1 + F\Gamma_{22}^2 = F_v - \frac{1}{2}G_u,$$
$$F\Gamma_{11}^1 + G\Gamma_{11}^2 = F_u - \frac{1}{2}E_v, \qquad F\Gamma_{12}^1 + G\Gamma_{12}^2 = \frac{1}{2}G_u, \qquad F\Gamma_{22}^1 + G\Gamma_{22}^2 = \frac{1}{2}G_v.$$

Plugging in $E = G = 1$ and $F = \cos(uv)$, we obtain

$$\Gamma^1_{11} + \cos(uv)\Gamma^2_{11} = 0, \qquad \Gamma^1_{12} + \cos(uv)\Gamma^2_{12} = 0, \qquad \Gamma^1_{22} + \cos(uv)\Gamma^2_{22} = -u\sin(uv),$$
$$\cos(uv)\Gamma^1_{11} + \Gamma^2_{11} = -v\sin(uv), \qquad \cos(uv)\Gamma^1_{12} + \Gamma^2_{12} = 0, \qquad \cos(uv)\Gamma^1_{22} + \Gamma^2_{22} = 0.$$

From this one *could* easily obtain that

$$\Gamma^1_{11} = \frac{v\cos(uv)}{\sin(uv)}, \qquad \Gamma^1_{12} = 0, \qquad \Gamma^1_{22} = -\frac{u}{\sin(uv)},$$
$$\Gamma^2_{11} = -\frac{v}{\sin(uv)}, \qquad \Gamma^2_{12} = 0, \qquad \Gamma^2_{22} = \frac{u\cos(uv)}{\sin(uv)}.$$

However, we will see now that we can avoid computations of Γ^1_{22} and Γ^2_{22}.

(b) *Step 2: Calculate $LN - M^2$:*
From the equations above we have

$$\begin{aligned}
LN &= (L\boldsymbol{N}) \cdot (N\boldsymbol{N}) \\
&= (\boldsymbol{x}_{uu} - \Gamma^1_{11}\boldsymbol{x}_u - \Gamma^2_{11}\boldsymbol{x}_v) \cdot (\boldsymbol{x}_{vv} - \Gamma^1_{22}\boldsymbol{x}_u - \Gamma^2_{22}\boldsymbol{x}_v) \\
&= \boldsymbol{x}_{uu} \cdot \boldsymbol{x}_{vv} - \Gamma^1_{22} \underbrace{\boldsymbol{x}_{uu} \cdot \boldsymbol{x}_u}_{=0} - \Gamma^2_{22} \underbrace{\boldsymbol{x}_{uu} \cdot \boldsymbol{x}_v}_{=-v\sin(uv)} - \Gamma^1_{11} \underbrace{\boldsymbol{x}_{vv} \cdot \boldsymbol{x}_u}_{=-u\sin(uv)} - \Gamma^2_{11} \underbrace{\boldsymbol{x}_{vv} \cdot \boldsymbol{x}_v}_{=0} \\
&\qquad + \Gamma^1_{11}\Gamma^1_{22}E + (\Gamma^1_{11}\Gamma^2_{22} + \Gamma^2_{11}\Gamma^1_{22})F + \Gamma^2_{11}\Gamma^2_{22}G \\
&= \boldsymbol{x}_{uu} \cdot \boldsymbol{x}_{vv} + (\Gamma^2_{22}v + \Gamma^1_{11}u)\sin(uv) \\
&\qquad + \Gamma^1_{11}\Gamma^1_{22} + (\Gamma^1_{11}\Gamma^2_{22} + \Gamma^2_{11}\Gamma^1_{22})\cos(uv) + \Gamma^2_{11}\Gamma^2_{22} \\
&= \boldsymbol{x}_{uu} \cdot \boldsymbol{x}_{vv} + \Gamma^2_{22}\big(\Gamma^1_{11} + \Gamma^2_{11} + v\sin(uv)\big) + \Gamma^1_{22}\big(\Gamma^1_{11} + \Gamma^2_{11}\cos(uv)\big) + \Gamma^1_{11}u\sin(uv).
\end{aligned}$$

Note that due to the defining equations on Γ^1_{11} and Γ^2_{11} the first and second parentheses in the expression above vanish, i.e.,

$$LN = \boldsymbol{x}_{uu} \cdot \boldsymbol{x}_{vv} + \Gamma^1_{11}u\sin(uv),$$

which means that all we needed is to calculate $\Gamma^1_{11} = \frac{v\cos(uv)}{\sin(uv)}$.
Let us now calculate M^2. First we observe that the linear system involving Γ^1_{12} and Γ^2_{12} is homogeneous with non-zero determinant, so has a trivial solution only, i.e. $\Gamma^1_{12} = 0 = \Gamma^2_{12}$. Hence,

$$\begin{aligned}
M^2 &= (M\boldsymbol{N}) \cdot (M\boldsymbol{N}) \\
&= (\boldsymbol{x}_{uv} - \Gamma^1_{12}\boldsymbol{x}_u - \Gamma^2_{12}\boldsymbol{x}_v) \cdot (\boldsymbol{x}_{vv} - \Gamma^1_{12}\boldsymbol{x}_u - \Gamma^2_{12}\boldsymbol{x}_v) \\
&= \boldsymbol{x}_{uv} \cdot \boldsymbol{x}_{uv},
\end{aligned}$$

so that

$$LN - M^2 = \boldsymbol{x}_{uu} \cdot \boldsymbol{x}_{vv} - \boldsymbol{x}_{uv} \cdot \boldsymbol{x}_{uv} + \Gamma^1_{11}u\sin(uv) = \boldsymbol{x}_{uu} \cdot \boldsymbol{x}_{vv} - \boldsymbol{x}_{uv} \cdot \boldsymbol{x}_{uv} + uv\cos(uv).$$

Finally, recall that

$$\begin{aligned}
\boldsymbol{x}_{uu} \cdot \boldsymbol{x}_{vv} - \boldsymbol{x}_{uv} \cdot \boldsymbol{x}_{uv} &= (\boldsymbol{x}_u \cdot \boldsymbol{x}_{vv})_u - (\boldsymbol{x}_u \cdot \boldsymbol{x}_{uv})_v \\
&= \big(F_v - G_u/2\big)_u - (E_v/2)_v \\
&= (-u\sin(uv))_u - 0 \\
&= -\sin(uv) - uv\cos(uv),
\end{aligned}$$

so that finally,

$$\begin{aligned}
LN - M^2 &= -\sin(uv) - uv\cos(uv) + uv\cos(uv) \\
&= -\sin(uv).
\end{aligned}$$

(c) *Step 3: Calculate K:*
The Gauss curvature is

$$K = \frac{LN - M^2}{EG - F^2} = \frac{LN - M^2}{1 - \cos^2(uv)} = \frac{-1}{\sin(uv)}$$

as desired.

5.3. ($\star$) If the coefficients of the first fundamental form of a surface S are given by

$$E = 2 + v^2, \qquad F = 1, \qquad G = 1,$$

show that the Gauss curvature of S is given by

$$K = -\frac{1}{(1+v^2)^2}.$$

Solution:

Calculations are similar to the previous exercise.

(a) *Step 1: Christoffel symbols Γ_{ij}^k.*
We have

$$E\Gamma_{11}^1 + F\Gamma_{11}^2 = \frac{1}{2}E_u, \qquad E\Gamma_{12}^1 + F\Gamma_{12}^2 = \frac{1}{2}E_v, \qquad E\Gamma_{22}^1 + F\Gamma_{22}^2 = F_v - \frac{1}{2}G_u,$$
$$F\Gamma_{11}^1 + G\Gamma_{11}^2 = F_u - \frac{1}{2}E_v, \qquad F\Gamma_{12}^1 + G\Gamma_{12}^2 = \frac{1}{2}G_u, \qquad F\Gamma_{22}^1 + G\Gamma_{22}^2 = \frac{1}{2}G_v.$$

Plugging in $E = 2 + v^2$ and $F = G = 1$, we obtain

$$(2+v^2)\Gamma_{11}^1 + \Gamma_{11}^2 = 0, \qquad (2+v^2)\Gamma_{12}^1 + \Gamma_{12}^2 = v, \qquad (2+v^2)\Gamma_{22}^1 + \Gamma_{22}^2 = 0,$$
$$\Gamma_{11}^1 + \Gamma_{11}^2 = -v, \qquad \Gamma_{12}^1 + \Gamma_{12}^2 = 0, \qquad \Gamma_{22}^1 + \Gamma_{22}^2 = 0.$$

We see that the equations on Γ_{22}^1 and Γ_{22}^2 have only the trivial solution $\Gamma_{22}^1 = \Gamma_{22}^2 = 0$. For the others, we obtain

$$\Gamma_{11}^1 = \frac{v}{(1+v^2)}, \qquad \Gamma_{11}^2 = -\frac{2v+v^3}{1+v^2}, \qquad \Gamma_{12}^1 = \frac{v}{1+v^2}, \qquad \Gamma_{12}^2 = -\frac{v}{1+v^2}.$$

(b) *Step 2: Calculate $LN - M^2$:*
We have

$$\begin{aligned}
LN &= (L\boldsymbol{N}) \cdot (N\boldsymbol{N}) \\
&= (\boldsymbol{x}_{uu} - \Gamma_{11}^1\boldsymbol{x}_u - \Gamma_{11}^2\boldsymbol{x}_v) \cdot (\boldsymbol{x}_{vv} - \Gamma_{22}^1\boldsymbol{x}_u - \Gamma_{22}^2\boldsymbol{x}_v) \\
&= (\boldsymbol{x}_{uu} - \Gamma_{11}^1\boldsymbol{x}_u - \Gamma_{11}^2\boldsymbol{x}_v) \cdot \boldsymbol{x}_{vv} \\
&= \boldsymbol{x}_{uu} \cdot \boldsymbol{x}_{vv} - \Gamma_{11}^1\boldsymbol{x}_{vv} \cdot \boldsymbol{x}_u - \Gamma_{11}^2\boldsymbol{x}_{vv} \cdot \boldsymbol{x}_v
\end{aligned}$$

So we only need $\boldsymbol{x}_{vv} \cdot \boldsymbol{x}_v = G_v/2 = 0$ and $\boldsymbol{x}_{vv} \cdot \boldsymbol{x}_u = F_v - G_u/2 = 0$ in our case here, hence we have

$$LN = \boldsymbol{x}_{uu} \cdot \boldsymbol{x}_{vv}.$$

Similarly, for M^2 we obtain

$$\begin{aligned}
M^2 &= (M\boldsymbol{N}) \cdot (M\boldsymbol{N}) \\
&= (\boldsymbol{x}_{uv} - \Gamma_{12}^1\boldsymbol{x}_u - \Gamma_{12}^2\boldsymbol{x}_v) \cdot (\boldsymbol{x}_{vv} - \Gamma_{22}^1\boldsymbol{x}_u - \Gamma_{22}^2\boldsymbol{x}_v) \\
&= \boldsymbol{x}_{uv} \cdot \boldsymbol{x}_{uv} - 2\Gamma_{12}^1 \underbrace{\boldsymbol{x}_{uv} \cdot \boldsymbol{x}_u}_{=E_v/2=v} - 2\Gamma_{12}^2 \underbrace{\boldsymbol{x}_{uv} \cdot \boldsymbol{x}_v}_{=G_u/2=0} \\
&\qquad + (\Gamma_{12}^1)^2 \underbrace{\boldsymbol{x}_u \cdot \boldsymbol{x}_u}_{=E=2+v^2} + 2\Gamma_{12}^1\Gamma_{12}^2 \underbrace{\boldsymbol{x}_u \cdot \boldsymbol{x}_v}_{=F=1} + (\Gamma_{12}^2)^2 \underbrace{\boldsymbol{x}_v \cdot \boldsymbol{x}_v}_{=G=1} \\
&= \boldsymbol{x}_{uv} \cdot \boldsymbol{x}_{uv} - \frac{2v^2}{1+v^2} \\
&\qquad + \frac{v^2(2+v^2)}{(1+v^2)^2} - \frac{2v^2}{(1+v^2)^2} + \frac{v^2}{(1+v^2)^2} \\
&= \boldsymbol{x}_{uv} \cdot \boldsymbol{x}_{uv} - \frac{2v^2}{1+v^2} \\
&\qquad + \frac{v^2(1+v^2)}{(1+v^2)^2} \\
&= \boldsymbol{x}_{uv} \cdot \boldsymbol{x}_{uv} - \frac{v^2}{1+v^2}
\end{aligned}$$

Hence

$$LN - M^2 = \boldsymbol{x}_{uu} \cdot \boldsymbol{x}_{vv} - \boldsymbol{x}_{uv} \cdot \boldsymbol{x}_{uv} + \frac{v^2}{1+v^2}.$$

Recall again that

$$\begin{aligned} \boldsymbol{x}_{uu} \cdot \boldsymbol{x}_{vv} - \boldsymbol{x}_{uv} \cdot \boldsymbol{x}_{uv} &= (\boldsymbol{x}_u \cdot \boldsymbol{x}_{vv})_u - (\boldsymbol{x}_u \cdot \boldsymbol{x}_{uv})_v \\ &= \left(F_v - \frac{1}{2}G_u\right)_u - \left(\frac{1}{2}E_v\right)_v \\ &= -1, \end{aligned}$$

so that finally

$$LN - M^2 = -1 + \frac{v^2}{1+v^2} = -\frac{1}{1+v^2}$$

(c) *Step 3: Calculate K:*
The Gauss curvature is

$$K = \frac{LN - M^2}{EG - F^2} = \frac{-1/(1+v^2)}{(2+v^2) - 1} = \frac{-1}{(1+v^2)^2}$$

as desired.

5.4. Let $\boldsymbol{x}$ be a local parametrization of a surface S such that $E = 1$, $F = 0$ and G is a function of u only. Show that

$$\Gamma^2_{12} = \Gamma^2_{21} = \frac{G_u}{2G}, \quad \Gamma^1_{22} = -\frac{G_u}{2}$$

and that all the other Christoffel symbols are zero. Hence show that the Gauss curvature K of S is given by

$$K = -\frac{(\sqrt{G})_{uu}}{\sqrt{G}}.$$

Solution:

Again, the calculations are similar to the previous exercise.

(a) *Step 1: Christoffel symbols Γ^k_{ij}.*
Since $E = 1$, $F = 0$ and $G_v = 0$, we have

$$\begin{aligned} &\Gamma^1_{11} = 0, && \Gamma^1_{12} = 0, && \Gamma^1_{22} = -\frac{1}{2}G_u, \\ &G\Gamma^2_{11} = 0, && G\Gamma^2_{12} = \frac{1}{2}G_u, && G\Gamma^2_{22} = 0, \end{aligned}$$

which implies

$$(\Gamma^2_{21} =)\Gamma^2_{12} = \frac{G_u}{2G} \qquad \text{and} \qquad \Gamma^1_{22} = -\frac{G_u}{2},$$

and all other Christoffel symbols are 0, as desired (note that G cannot vanish as the first fundamental form is positive definite).

(b) *Step 2: Calculate $LN - M^2$.*
We have

$$\begin{aligned} LN &= (L\boldsymbol{N}) \cdot (N\boldsymbol{N}) \\ &= (\boldsymbol{x}_{uu} - \Gamma^1_{11}\boldsymbol{x}_u - \Gamma^2_{11}\boldsymbol{x}_v) \cdot (\boldsymbol{x}_{vv} - \Gamma^1_{22}\boldsymbol{x}_u - \Gamma^2_{22}\boldsymbol{x}_v) \\ &= \boldsymbol{x}_{uu} \cdot (\boldsymbol{x}_{vv} - \Gamma^1_{22}\boldsymbol{x}_u) \\ &= \boldsymbol{x}_{uu} \cdot \boldsymbol{x}_{vv} - \Gamma^1_{22} \underbrace{\boldsymbol{x}_{uu} \cdot \boldsymbol{x}_u}_{=E_u/2=0} \\ &= \boldsymbol{x}_{uu} \cdot \boldsymbol{x}_{vv}. \end{aligned}$$

Similarly, for M^2 we obtain

$$\begin{aligned}
M^2 &= (M\boldsymbol{N})\cdot(M\boldsymbol{N}) \\
&= (\boldsymbol{x}_{uv} - \Gamma_{12}^1\boldsymbol{x}_u - \Gamma_{12}^2\boldsymbol{x}_v)\cdot(\boldsymbol{x}_{vv} - \Gamma_{22}^1\boldsymbol{x}_u - \Gamma_{22}^2\boldsymbol{x}_v) \\
&= (\boldsymbol{x}_{uv} - \Gamma_{12}^2\boldsymbol{x}_v)\cdot(\boldsymbol{x}_{vv} - \Gamma_{12}^2\boldsymbol{x}_v) \\
&= \boldsymbol{x}_{uv}\cdot\boldsymbol{x}_{uv} - 2\Gamma_{12}^2\underbrace{\boldsymbol{x}_{uv}\cdot\boldsymbol{x}_v}_{=G_u/2} + (\Gamma_{12}^2)^2\underbrace{\boldsymbol{x}_v\cdot\boldsymbol{x}_v}_{=G} \\
&= \boldsymbol{x}_{uv}\cdot\boldsymbol{x}_{uv} - \frac{(G_u)^2}{2G} + \frac{(G_u)^2}{4G} = \boldsymbol{x}_{uv}\cdot\boldsymbol{x}_{uv} - \frac{(G_u)^2}{4G}
\end{aligned}$$

Hence,

$$LN - M^2 = \boldsymbol{x}_{uu}\cdot\boldsymbol{x}_{vv} - \boldsymbol{x}_{uv}\cdot\boldsymbol{x}_{uv} + \frac{(G_u)^2}{4G}$$

Recall again that

$$\begin{aligned}
\boldsymbol{x}_{uu}\cdot\boldsymbol{x}_{vv} - \boldsymbol{x}_{uv}\cdot\boldsymbol{x}_{uv} &= (\boldsymbol{x}_u\cdot\boldsymbol{x}_{vv})_u - (\boldsymbol{x}_u\cdot\boldsymbol{x}_{uv})_v \\
&= \Big(F_v - \frac{1}{2}G_u\Big)_u - \Big(\frac{1}{2}E_v\Big)_v \\
&= -\frac{1}{2}G_{uu},
\end{aligned}$$

so that finally

$$LN - M^2 = -\frac{1}{2}G_{uu} + \frac{(G_u)^2}{4G}.$$

(c) *Step 3: Calculate K.*
The Gauss curvature is

$$K = \frac{LN - M^2}{EG - F^2} = \frac{-G_{uu}/2 + (G_u)^2/4G}{G} = -\frac{G_{uu}}{2G} + \frac{(G_u)^2}{4G^2}.$$

Finally, we have

$$\frac{(\sqrt{G})_{uu}}{\sqrt{G}} = \frac{1}{\sqrt{G}}\Big(\frac{G_u}{2\sqrt{G}}\Big)_u = \frac{1}{\sqrt{G}}\Big(\frac{G_{uu}}{2\sqrt{G}} - \frac{(G_u)^2}{4G^{3/2}}\Big) = \frac{G_{uu}}{2G} - \frac{(G_u)^2}{4G^2} = -K$$

so that we obtain the desired formula.

Remark. A particular example of such coefficients is given by a surface of revolution with a generating curve parametrized by arc length with u and v interchanged.