Durham University
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Differential Geometry III, Solutions 6 (Week 16)

Curves on surfaces

6.1. Let $\{\boldsymbol{e}_1, \boldsymbol{e}_2\}$ be an orthonormal basis of T_pS consisting of eigenvectors of the Weingarten map $-d_p\boldsymbol{N}$ with corresponding eigenvalues κ_1, κ_2. If $\boldsymbol{e} = (\cos\vartheta)\boldsymbol{e}_1 + (\sin\vartheta)\boldsymbol{e}_2$, show, that the normal curvature κ_{n} of a curve tangential to $\boldsymbol{e}$ is given by

$$\kappa_{\mathrm{n}}(\vartheta) = \kappa_1 \cos^2\vartheta + \kappa_2 \sin^2\vartheta.$$

Deduce that

$$\frac{1}{2\pi}\int_0^{2\pi} \kappa_{\mathrm{n}}(\vartheta)\,\mathrm{d}\vartheta = H,$$

where H denotes the mean curvature of S at p. (This justifies the term *mean curvature*).

Solution:

Note first that

$$I_p(\boldsymbol{e}) = \|\boldsymbol{e}\|^2 = \big\|(\cos\vartheta)\boldsymbol{e}_1 + (\sin\vartheta)\boldsymbol{e}_2\big\|^2 = \cos^2\vartheta + \sin^2\vartheta = 1$$

by Pythagoras' Theorem (as $\boldsymbol{e}_1$, $\boldsymbol{e}_2$ are orthonormal). The normal curvature κ_{n} of a curve with tangent vector $\boldsymbol{e}$ at p is given by

$$\begin{aligned}
\kappa_{\mathrm{n}}(\vartheta) &= \frac{I\!I_p(\boldsymbol{e})}{I_p(\boldsymbol{e})} = I\!I_p(\boldsymbol{e}) \\
&= I\!I_p\big((\cos\vartheta)\boldsymbol{e}_1 + (\sin\vartheta)\boldsymbol{e}_2\big) \\
&= -\big\langle d_p\boldsymbol{N}((\cos\vartheta)\boldsymbol{e}_1 + (\sin\vartheta)\boldsymbol{e}_2), (\cos\vartheta)\boldsymbol{e}_1 + (\sin\vartheta)\boldsymbol{e}_2\big\rangle \\
&= \big\langle \kappa_1(\cos\vartheta)\boldsymbol{e}_1 + \kappa_2(\sin\vartheta)\boldsymbol{e}_2), (\cos\vartheta)\boldsymbol{e}_1 + (\sin\vartheta)\boldsymbol{e}_2\big\rangle \\
&= \kappa_1\cos^2\vartheta + \kappa_2\sin^2\vartheta
\end{aligned}$$

by Meusnier's theorem (first equality), the definition of the second fundamental form (fourth equality), the fact that $\boldsymbol{e}_1$, $\boldsymbol{e}_2$ are eigenvectors of $-d_p\boldsymbol{N}$ (fifth equality) and that $\boldsymbol{e}_1$, $\boldsymbol{e}_2$ are orthonormal (last equality). This shows the first formula.

For the second, just note that

$$\frac{1}{2\pi}\int_0^{2\pi} \kappa_{\mathrm{n}}(\vartheta)\,\mathrm{d}\vartheta = \frac{1}{2\pi}\int_0^{2\pi}(\kappa_1\cos^2\vartheta + \kappa_2\sin^2\vartheta)\,\mathrm{d}\vartheta = \frac{1}{2\pi}(\kappa_1\pi + \kappa_2\pi) = \frac{1}{2}(\kappa_1+\kappa_2) = H(p)$$

as $\int_0^{2\pi}\cos^2\vartheta\,\mathrm{d}\vartheta = \pi$ and similarly for the integral over $\sin^2\vartheta$.

6.2. Let $\boldsymbol{\alpha}$ be the curve defined by

$$\boldsymbol{\alpha}(t) = \mathrm{e}^t(\cos t, \sin t, 1) \qquad \text{for } t\in\mathbb{R}.$$

Observe that $\boldsymbol{\alpha}$ lies on the circular cone $S = \{\,(x,y,z)\in\mathbb{R}^3 \mid x^2+y^2 = z^2\,\}$.

Show that the normal curvature of $\boldsymbol{\alpha}$ in S is inversely proportional to e^t.

Solution:

Clearly,

$$(\mathrm{e}^t\cos t)^2 + (\mathrm{e}^t\sin t)^2 = (\mathrm{e}^t)^2,$$

so $\boldsymbol{\alpha}(t)\in S$ for all $t\in\mathbb{R}$. For further purposes, we also need

$$\boldsymbol{\alpha}'(t) = \mathrm{e}^t(\cos t - \sin t, \sin t + \cos t, 1).$$

Calculation of the normal curvature — reparametrization by arc length: Since $\|\boldsymbol{\alpha}'(t)\| = \sqrt{3}\mathrm{e}^t$ we set

$$s = \int_{-\infty}^{t} \sqrt{3}\mathrm{e}^u \,\mathrm{d}u = \sqrt{3}\mathrm{e}^t$$

so that $t = \log(s/\sqrt{3}) = \log s - (\log 3)/2$. Let us now call the reparametrized curve $\boldsymbol{\beta}$, i.e., we set

$$\boldsymbol{\beta}(s) = \boldsymbol{\alpha}(\log(s/\sqrt{3})) = \frac{s}{\sqrt{3}}\Big(\cos\log\frac{s}{\sqrt{3}}, \sin\log\frac{s}{\sqrt{3}}, 1\Big)$$

and therefore, we have

$$\begin{aligned}
\boldsymbol{\beta}'(s) &= \frac{1}{\sqrt{3}}\Big(\cos\log\frac{s}{\sqrt{3}} - \sin\log\frac{s}{\sqrt{3}}, \sin\log\frac{s}{\sqrt{3}} + \cos\log\frac{s}{\sqrt{3}}, 1\Big),\\
\boldsymbol{\beta}''(s) &= \frac{1}{s\sqrt{3}}\Big(-\sin\log\frac{s}{\sqrt{3}} - \cos\log\frac{s}{\sqrt{3}}, \cos\log\frac{s}{\sqrt{3}} - \sin\log\frac{s}{\sqrt{3}}, 0\Big)
\end{aligned}$$

How can we efficiently calculate the normal vector for a surface defined by an equation? At $p = (x, y, z)$, for a surface $S = \{\,(x, y, z) \mid f(x, y, z) = 0\,\}$ we have (here with $f(x, y, z) = x^2 + y^2 - z^2$, hence $\nabla f(x, y, z) = 2(x, y, -z)$)

$$\boldsymbol{N}(p) = \frac{1}{\|\nabla f(p)\|}\nabla f(p) = \frac{1}{\sqrt{x^2 + y^2 + z^2}}(x, y, -z) \ \Big(= \frac{1}{\|p\|}(x, y, -z) \text{ here. }\Big)$$

— So there is no need to find a parametrization and then calculate $\boldsymbol{x}_u \times \boldsymbol{x}_v$ etc. —

Now, the normal curvature of the curve $\boldsymbol{\beta}$ (and hence $\boldsymbol{\alpha}$) is

$$\begin{aligned}
\kappa_{\mathrm{n},\boldsymbol{\beta}}(s) &= \boldsymbol{\beta}''(s) \cdot \boldsymbol{N}(\boldsymbol{\beta}(s))\\
&= \frac{1}{s\sqrt{3}\|\boldsymbol{\beta}(s)\|}\Big(\Big(-\sin\log\frac{s}{\sqrt{3}} - \cos\log\frac{s}{\sqrt{3}}\Big)\frac{s}{\sqrt{3}}\cos\log\frac{s}{\sqrt{3}}\\
&\qquad + \Big(\cos\log\frac{s}{\sqrt{3}} - \sin\log\frac{s}{\sqrt{3}}\Big)\frac{s}{\sqrt{3}}\sin\log\frac{s}{\sqrt{3}}\Big)\\
&= \frac{-1}{3\|\boldsymbol{\beta}(s)\|}\Big(\cos^2\log\frac{s}{\sqrt{3}} + \sin^2\log\frac{s}{\sqrt{3}}\Big) = -\frac{1}{3\|\boldsymbol{\beta}(s)\|}
\end{aligned}$$

and since

$$\|\boldsymbol{\beta}(s)\|^2 = \frac{s^2}{3}\Big(\cos^2\log\frac{s}{\sqrt{3}} + \sin^2\log\frac{s}{\sqrt{3}} + 1\Big) = \frac{2s^2}{3},$$

we have $\kappa_{\mathrm{n},\boldsymbol{\beta}}(s) = -1/(3\sqrt{2s^2/3}) = -(\sqrt{3/2})/(3s)$ and finally

$$\kappa_{\mathrm{n},\boldsymbol{\alpha}}(t) = \kappa_{\mathrm{n},\boldsymbol{\beta}}(\sqrt{3}\mathrm{e}^t) = -\sqrt{\frac{3}{2}} \cdot \frac{1}{3\sqrt{3}\mathrm{e}^t} = -\frac{1}{3\sqrt{2}\mathrm{e}^t}$$

which is inversely proportional to e^t as desired.

Alternative approach: Calculation of the normal curvature using a local parametrization. If we parametrize the surface S as a surface of revolution by

$$\boldsymbol{x}(u, v) = (v\cos u, v\sin u, v), \qquad (u, v) \in (-\pi, \pi) \times (0, \infty) \text{ or } (u, v) \in (0, 2\pi) \times (0, \infty)$$

then $\boldsymbol{\alpha}$ is given in these parametrization as

$$\boldsymbol{\alpha}(t) = (\mathrm{e}^t\cos t, \mathrm{e}^t\sin t, \mathrm{e}^t)\boldsymbol{x}(u(t), v(t))$$

which means that

$$u(t) = t \qquad \text{and} \qquad v(t) = \mathrm{e}^t.$$

Now, the formula for the normal curvature of $\boldsymbol{\alpha}$ in a local parametrization is given by

$$\kappa_{\rm n} = \frac{(u')^2L + 2u'v'M + (v')^2N}{(u')^2E + 2u'v'F + (v')^2G},$$

so we need the coefficients of the first and second fundamental form. Since

$$\boldsymbol{x}_u = (-v\sin u, v\cos u, 0) \qquad \text{and} \qquad \boldsymbol{x}_v = (\cos u, \sin u, 1),$$

we have $\boldsymbol{x}_u \times \boldsymbol{x}_v = v(\cos u, \sin u, -1)$, hence

$$\boldsymbol{N} = \frac{1}{\sqrt{2}}(\cos u, \sin u, -1)$$

and $E(u,v) = v^2$, $F = 0$ and $G = 2$. Moreover,

$$\boldsymbol{x}_{uu} = (-v\cos u, -v\sin u, 0), \qquad \boldsymbol{x}_{uv} = (-\sin u, \cos u, 0), \qquad \boldsymbol{x}_{vv} = (0,0,0),$$

so that

$$L = \boldsymbol{x}_{uu}\cdot\boldsymbol{N} = -\frac{v}{\sqrt{2}}, \qquad M = 0, \qquad N = 0.$$

Moreover, $u'(t) = 1$ and $v'(t) = \mathrm{e}^t$, so that finally

$$\kappa_{\rm n} = \frac{-(u')^2v/\sqrt{2}}{(u')^2v^2 + 2(v')^2} = \frac{-\mathrm{e}^t/\sqrt{2}}{\mathrm{e}^{2t} + 2\mathrm{e}^{2t}} = -\frac{1}{\mathrm{e}^t 3\sqrt{2}}$$

6.3. Show that an asymptotic curve can only exist in the hyperbolic or flat region $\{\, p \in S \mid K(p) \le 0 \,\}$. (In other words, if a surface is elliptic everywhere, then there is no asymptotic curve.)

Solution:

A curve is an asymptotic curve iff $I\!I_{\boldsymbol{\alpha}(s)}(\alpha') = 0$. If $K(p) > 0$, then $LN - M^2 > 0$, which implies that the second fundamental form is either positive definite or negative definite (recall the Krammer's rule), any of these implies that $I\!I_{\boldsymbol{\alpha}(s)}$ never takes zero values.

6.4. Let S be a surface in $\mathbb{R}^3$ with Gauss map $\boldsymbol{N}$, and let $\boldsymbol{\beta}$ be a regular curve on S not necessarily parametrized by arc length. Show that the geodesic curvature $\kappa_{\rm g}$ of $\boldsymbol{\beta}$ is given by

$$\kappa_{\rm g} = \frac{1}{\|\boldsymbol{\beta}'\|^3}(\boldsymbol{\beta}' \times \boldsymbol{\beta}'')\cdot\boldsymbol{N}.$$

Solution:

Assume that $\boldsymbol{\beta}\colon [t_0, t_1] \longrightarrow S$ is the parametrization of the curve. Let us first parametrize the curve by arc length, i.e., set

$$s = \varphi(t) := \int_{t_0}^{t} \|\boldsymbol{\beta}'(u)\|\,\mathrm{d}u,$$

then $\mathrm{d}s/\mathrm{d}t = \varphi'(t) = \|\boldsymbol{\beta}'(t)\|$ and we set

$$\boldsymbol{\alpha} := \boldsymbol{\beta}\circ\varphi^{-1}, \qquad \text{i.e.} \qquad \boldsymbol{\alpha}(s) := \boldsymbol{\beta}(\varphi^{-1}(s)) = \boldsymbol{\beta}(t)$$

if $t = \varphi(s)$. Clearly (as we did in the first term),

$$\boldsymbol{\alpha}'(s) = (\varphi^{-1})'(s)\boldsymbol{\beta}'(\varphi^{-1}(s)) = \frac{1}{\|\boldsymbol{\beta}'(t)\|}\boldsymbol{\beta}'(t)$$

since $(\varphi^{-1})'(s) = 1/\varphi'(t) = 1/\|\boldsymbol{\beta}'(t)\|$ which we can also write formally as

$$\frac{\mathrm{d}}{\mathrm{d}s} = \frac{1}{\|\boldsymbol{\beta}'(t)\|}\frac{\mathrm{d}}{\mathrm{d}t}.$$

Moreover,

$$\begin{aligned}\boldsymbol{\alpha}''(s) = \frac{\mathrm{d}}{\mathrm{d}s}\Big(\frac{1}{\|\boldsymbol{\beta}'(t)\|}\boldsymbol{\beta}'(t)\Big) &= \frac{1}{\|\boldsymbol{\beta}'(t)\|}\frac{\mathrm{d}}{\mathrm{d}t}\Big(\frac{1}{\|\boldsymbol{\beta}'(t)\|}\boldsymbol{\beta}'(t)\Big)\\ &= \underbrace{\frac{1}{\|\boldsymbol{\beta}'(t)\|}\frac{\mathrm{d}}{\mathrm{d}t}\Big(\frac{1}{\|\boldsymbol{\beta}'(t)\|}\Big)\boldsymbol{\beta}'(t)}_{\text{proportional to } \boldsymbol{\alpha}'} + \frac{1}{\|\boldsymbol{\beta}'(t)\|^2}\boldsymbol{\beta}''(t).\end{aligned}$$

Let now $\boldsymbol{N}$ be the normal to the surface (at $\boldsymbol{\alpha}(s) = \boldsymbol{\beta}(t)$). We have

$$\begin{aligned}\kappa_{\mathrm{g}}(s) = \boldsymbol{\alpha}''(s)\cdot(\boldsymbol{N}(\boldsymbol{\alpha}(s))\times\boldsymbol{\alpha}'(s)) &= \frac{1}{\|\boldsymbol{\beta}'(t)\|^2}\boldsymbol{\beta}''(t)\cdot(\boldsymbol{N}(\boldsymbol{\alpha}(s))\times\boldsymbol{\alpha}'(s))\\ &= \frac{1}{\|\boldsymbol{\beta}'(t)\|^3}\boldsymbol{\beta}''(t)\cdot(\boldsymbol{N}(\boldsymbol{\beta}(t))\times\boldsymbol{\beta}'(t))\\ &= \frac{1}{\|\boldsymbol{\beta}'(t)\|^3}(\boldsymbol{\beta}'(t)\times\boldsymbol{\beta}''(t))\cdot\boldsymbol{N}(\boldsymbol{\beta}(t))\end{aligned}$$

as $\boldsymbol{\beta}'$ is proportional to $\boldsymbol{\alpha}'$, hence orthogonal to $\boldsymbol{N}\times\boldsymbol{\alpha}'$ (for the second equality) and where we used

$$\boldsymbol{b}\cdot(\boldsymbol{c}\times\boldsymbol{a}) = \boldsymbol{c}\cdot(\boldsymbol{a}\times\boldsymbol{b}) = (\boldsymbol{a}\times\boldsymbol{b})\cdot\boldsymbol{c}.$$

In particular, we have shown the desired formula.

6.5. Let S be Enneper's surface (see Problem 4.2) parametrized by

$$\boldsymbol{x}(u,v) = \Big(u - \frac{u^3}{3} + uv^2, v - \frac{v^3}{3} + u^2v, u^2 - v^2\Big), \qquad (u,v)\in\mathbb{R}^2.$$

(a) Calculate the lines of curvature.

(b) Show that the asymptotic curves are given by $u \pm v = \text{const}$.

Solution:

We have calculated the coefficients of the first and second fundamental form w.r.t. $\boldsymbol{x}$ in Problem 4.2 as

$$E(u,v) = G(u,v) = (1+u^2+v^2)^2,\ F(u,v) = 0 \qquad \text{and} \qquad L = 2,\ M = 0,\ N = -2;$$

(a) Since the parametrization if *principal* (i.e., $F = 0$ and $M = 0$), the lines of curvature are just the coordinate curves (see, e.g., Prop. 11.18, or this can be easily computed explicitly). Hence they are given by $s \mapsto \boldsymbol{x}(s, v_0)$ and $s \mapsto \boldsymbol{x}(u_0, s)$ for u_0, $v_0 \in \mathbb{R}$.

(b) A curve $\boldsymbol{\alpha}$ with local parametrization $\boldsymbol{\alpha}(s) = \boldsymbol{x}(u(s), v(s))$ is an *asymptotic curve* if $\kappa_{\mathrm{n}} = 0$, i.e., if $I\!I_{\boldsymbol{\alpha}(s)}(\boldsymbol{\alpha}'(s) = 0$, or,

$$(u')^2L + 2u'v'M + (v')2N = 0$$

Here it means that

$$2(u')^2 - 2(v')^2 = 2(u'+v')(u'-v') = 0 \qquad \text{or, equivalently} \qquad (u-v)' = 0 \text{ or } (u+v)' = 0,$$

which is equivalent to $u \pm v = \text{const}$.

6.6. (a) ($\star$) Show that the asymptotic curves on the surface given by $x^2 + y^2 - z^2 = 1$ are straight lines.

(b) Let S be a ruled surface. What are necessary and sufficient assumptions on S for all asymptotic curves being straight lines?

Hint: use linear algebra.

Solution:

(a) The surface is a one-sheeted hyperboloid, so it is doubly ruled (i.e. there are two lines through every point). As we have already proved, all lines are asymptotic curves, so we only need to prove that there are no others.

If $\{\boldsymbol{e}_1, \boldsymbol{e}_2\}$ is a basis of T_pS consisting of eigenvectors of $-d_p\boldsymbol{N}$, then $II_p(\boldsymbol{e}_i) = \kappa_i$, where κ_i are principal curvatures, and $\langle \boldsymbol{e}_1, \boldsymbol{e}_2 \rangle = 0$ (there are no umbilic points since $K < 0$ everywhere). Therefore,

$$II_p(a\boldsymbol{e}_1 + b\boldsymbol{e}_2) = a^2\kappa_1 + b^2\kappa_2,$$

which vanishes in the only case when $b = \pm a\sqrt{-\kappa_1/\kappa_2}$, so there are exactly two directions on which II_p vanishes. This completes the proof.

Equivalently, we could say that any indefinite form of rank 2 looks like $x^2 - y^2$ in some basis, so there are two vectors with zero value only.

Alternatively, one could parametrize the hyperboloid as a ruled surface via

$$\boldsymbol{x}(u, v) = \big(\cos(u), \sin(u), 0) + v(\sin(u), -\cos(u), 1),$$

then compute the coefficients of the second fundamental form, solve the differential equation

$$(u')^2L(u, v) + 2u'v'M(u, v) + (v')^2N(u, v) = 0$$

and observe that the solutions will be precisely the lines.

(b) The proof of (a) can be applied to any doubly ruled surface, so for these surfaces indeed all the asymptotic curves are lines. The statement is obviously true for planes as well. Let us prove that for all other surfaces the statement does not hold.

So, assume that S is neither a plane nor a doubly ruled surface. As we have already seen above, since S is ruled all the points are either hyperbolic or flat, which means that there are no umbilic points (except for some isolated planar ones), and every point $p \in S$ has precisely two directions on which II_p vanishes, one of which is the direction of the ruling. Note that these lines do not intersect each other in a ruled surface, so we can take another asymptotic curve through every point which will not be a line (formally speaking here we use the theorem of existence of a solution of differential equation with given initial data).