Durham University
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Epiphany 2017

Differential Geometry III, Homework 6 (Week 16)

Due date for starred problems: **Thursday, March 2**.

Curves on surfaces

6.1. Let $\{\boldsymbol{e}_1, \boldsymbol{e}_2\}$ be an orthonormal basis of T_pS consisting of eigenvectors of the Weingarten map $-d_p\boldsymbol{N}$ with corresponding eigenvalues κ_1, κ_2. If $\boldsymbol{e} = (\cos\vartheta)\boldsymbol{e}_1 + (\sin\vartheta)\boldsymbol{e}_2$, show, that the normal curvature κ_{n} of a curve tangential to $\boldsymbol{e}$ is given by

$$\kappa_{\mathrm{n}}(\vartheta) = \kappa_1 \cos^2\vartheta + \kappa_2 \sin^2\vartheta.$$

Deduce that

$$\frac{1}{2\pi}\int_0^{2\pi} \kappa_{\mathrm{n}}(\vartheta)\,\mathrm{d}\vartheta = H,$$

where H denotes the mean curvature of S at p. (This justifies the term *mean curvature*).

6.2. Let $\boldsymbol{\alpha}$ be the curve defined by

$$\boldsymbol{\alpha}(t) = \varepsilon^t(\cos t, \sin t, 1) \qquad \text{for } t \in \mathbb{R}.$$

Observe that $\boldsymbol{\alpha}$ lies on the circular cone $S = \{\,(x,y,z) \in \mathbb{R}^3 \mid x^2 + y^2 = z^2\,\}$.

Show that the normal curvature of $\boldsymbol{\alpha}$ in S is inversely proportional to ε^t.

6.3. Show that an asymptotic curve can only exist in the hyperbolic or flat region $\{\,p \in S \mid K(p) \le 0\,\}$. (In other words, if a surface is elliptic everywhere, then there is no asymptotic curve.)

6.4. Let S be a surface in $\mathbb{R}^3$ with Gauss map $\boldsymbol{N}$, and let $\boldsymbol{\beta}$ be a regular curve on S not necessarily parametrized by arc length. Show that the geodesic curvature κ_{g} of $\boldsymbol{\beta}$ is given by

$$\kappa_{\mathrm{g}} = \frac{1}{\|\boldsymbol{\beta}'\|^3}(\boldsymbol{\beta}' \times \boldsymbol{\beta}'') \cdot \boldsymbol{N}.$$

6.5. Let S be Enneper's surface (see Problem 4.2) parametrized by

$$\boldsymbol{x}(u,v) = \Big(u - \frac{u^3}{3} + uv^2, v - \frac{v^3}{3} + u^2v, u^2 - v^2\Big), \qquad (u,v) \in \mathbb{R}^2.$$

(a) Calculate the lines of curvature.

(b) Show that the asymptotic curves are given by $u \pm v = \text{const}$.

6.6. (a) ($\star$) Show that the asymptotic curves on the surface given by $x^2 + y^2 - z^2 = 1$ are straight lines.

(b) Let S be a ruled surface. What are necessary and sufficient assumptions on S for all asymptotic curves being straight lines?

Hint: use linear algebra.