Durham University
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Differential Geometry III, Solutions 7 (Week 17)

Curves on surfaces. Geodesics.

7.1. If $\boldsymbol{x}$ is a local parametrization of a surface S in $\mathbb{R}^3$ with $E = 1$, $F = 0$ and G is a function of u only, write down the equations for $s \mapsto \boldsymbol{\alpha}(s) = \boldsymbol{x}(u(s), v(s))$ to be a geodesic. Conclude that the coordinate curves, where v is constant, are geodesics.

Solution:

The curve $\boldsymbol{\alpha}$ is a geodesic iff

$$u''E + \frac{1}{2}(u')^2E_u + u'v'E_v + (v')^2\Big(F_v - \frac{1}{2}G_u\Big) + v''F = 0$$
$$v''G + \frac{1}{2}(v')^2G_v + u'v'G_u + (u')^2\Big(F_u - \frac{1}{2}E_v\Big) + u''F = 0,$$

which reduces here to

$$u'' - \frac{1}{2}G_u(v')^2 = 0$$
$$v''G + u'v'G_u = 0.$$

Now, for a coordinate curve with v constant, we have $v' = 0$ and $v'' = 0$, so that the second equation is fulfilled. Moreover, the first one then becomes

$$u'' = 0.$$

Since the speed of $\boldsymbol{\alpha}$ is constant, we have

$$\|\boldsymbol{\alpha}'(s)\|^2 = (u')^2 + G(v')^2 = \text{const}.$$

Since $v' = 0$, we must have $u' \neq 0$ (otherwise $\boldsymbol{\alpha}'(s) = \boldsymbol{0}$), so that $u'' = 0$ as desired. Therefore $u(s) = u_0 + as$ (with $a \in \mathbb{R} \setminus \{0\}$) and the geodesic has the form

$$\boldsymbol{\alpha}(s) = \boldsymbol{x}(u_0 + as, v_0)$$

for some (u_0, v_0) in the parameter domain and some $a \in \mathbb{R}$.

7.2. Let $\boldsymbol{x}\colon U \longrightarrow S$ be a parametrization of a surface S, and let $\boldsymbol{\alpha}(s) = \boldsymbol{x}(u(s), v(s))$ be a curve parametrized by arc length. Find an expression for the geodesic curvature κ_{g} of $\boldsymbol{\alpha}$ involving u', v', u'', v'', E, F, G, Γ^i_{jk} (i.e. the *geodesic curvature is intrinsic*, κ_{g} depends only on the curve and the first fundamental form of the surface).

Solution:

The geodesic curvature is given by $\kappa_{\mathrm{g}} = \boldsymbol{\alpha}'' \cdot (\boldsymbol{N} \times \boldsymbol{\alpha}')$. Using the definition of the normal vector, and

$$\boldsymbol{\alpha}' = u'\boldsymbol{x}_u + v'\boldsymbol{x}_v$$

and its derivative

$$\boldsymbol{\alpha}'' = u''\boldsymbol{x}_u + (u')^2\boldsymbol{x}_{uu} + 2u'v'\boldsymbol{x}_{uv} + (v')^2\boldsymbol{x}_{vv} + v''\boldsymbol{x}_v$$

we obtain

$$\begin{aligned}
\kappa_g &= \boldsymbol{\alpha}'' \cdot (\boldsymbol{N} \times \boldsymbol{\alpha}') \\
&= \frac{1}{\|\boldsymbol{x}_u \times \boldsymbol{x}_v\|}\big(u''\boldsymbol{x}_u + (u')^2\boldsymbol{x}_{uu} + 2u'v'\boldsymbol{x}_{uv} + (v')^2\boldsymbol{x}_{vv} + v''\boldsymbol{x}_v\big) \cdot \big((\boldsymbol{x}_u \times \boldsymbol{x}_v) \times (u'\boldsymbol{x}_u + v'\boldsymbol{x}_v)\big) \\
&= \frac{1}{\|\boldsymbol{x}_u \times \boldsymbol{x}_v\|}\Big(v'u''\boldsymbol{x}_u \cdot ((\boldsymbol{x}_u \times \boldsymbol{x}_v) \times \boldsymbol{x}_v) \\
&\qquad + (u')^3\boldsymbol{x}_{uu} \cdot \big((\boldsymbol{x}_u \times \boldsymbol{x}_v) \times \boldsymbol{x}_u\big) + (u')^2v'\boldsymbol{x}_{uu} \cdot \big((\boldsymbol{x}_u \times \boldsymbol{x}_v) \times \boldsymbol{x}_v\big) \\
&\qquad + 2(u')^2v'\boldsymbol{x}_{uv} \cdot \big((\boldsymbol{x}_u \times \boldsymbol{x}_v) \times \boldsymbol{x}_u\big) + 2u'(v')^2\boldsymbol{x}_{uv} \cdot \big((\boldsymbol{x}_u \times \boldsymbol{x}_v) \times \boldsymbol{x}_v\big) \\
&\qquad + u'(v')^2\boldsymbol{x}_{vv} \cdot \big((\boldsymbol{x}_u \times \boldsymbol{x}_v) \times \boldsymbol{x}_u\big) + (v')^3\boldsymbol{x}_{vv} \cdot \big((\boldsymbol{x}_u \times \boldsymbol{x}_v) \times \boldsymbol{x}_v\big) \\
&\qquad + u'v''\boldsymbol{x}_v \cdot \big((\boldsymbol{x}_u \times \boldsymbol{x}_v) \times \boldsymbol{x}_u\big)\Big).
\end{aligned}$$

Note first that

$$\|\boldsymbol{x}_u \times \boldsymbol{x}_v\|^2 = EG - F^2$$

We now have to understand the expressions $\boldsymbol{x}_u \cdot ((\boldsymbol{x}_u \times \boldsymbol{x}_v) \times \boldsymbol{x}_v)$ etc. Using the rule

$$\boldsymbol{a} \times (\boldsymbol{b} \times \boldsymbol{c}) = (\boldsymbol{a} \cdot \boldsymbol{c})\boldsymbol{b} - (\boldsymbol{a} \cdot \boldsymbol{b})\boldsymbol{c} \quad \text{or, equivalently,} \quad (\boldsymbol{a} \times \boldsymbol{b}) \times \boldsymbol{c} = (\boldsymbol{a} \cdot \boldsymbol{c})\boldsymbol{b} - (\boldsymbol{b} \cdot \boldsymbol{c})\boldsymbol{a}$$

we obtain

$$\begin{aligned}
\boldsymbol{x}_u \cdot \big((\boldsymbol{x}_u \times \boldsymbol{x}_v) \times \boldsymbol{x}_v\big) &= \boldsymbol{x}_u \cdot \big((\boldsymbol{x}_u \cdot \boldsymbol{x}_v)\boldsymbol{x}_v - (\boldsymbol{x}_v \cdot \boldsymbol{x}_v)\boldsymbol{x}_u\big) = F^2 - EG \\
\boldsymbol{x}_{uu} \cdot \big((\boldsymbol{x}_u \times \boldsymbol{x}_v) \times \boldsymbol{x}_u\big) &= \boldsymbol{x}_{uu} \cdot \big((\boldsymbol{x}_u \cdot \boldsymbol{x}_u)\boldsymbol{x}_v - (\boldsymbol{x}_v \cdot \boldsymbol{x}_u)\boldsymbol{x}_u\big) = E\boldsymbol{x}_{uu} \cdot \boldsymbol{x}_v - F\boldsymbol{x}_{uu} \cdot \boldsymbol{x}_u \\
&= E\Big(F_u - \frac{1}{2}E_v\Big) - \frac{1}{2}FE_u \\
&= (EG - F^2)\Gamma^2_{11} \\
\boldsymbol{x}_{uu} \cdot \big((\boldsymbol{x}_u \times \boldsymbol{x}_v) \times \boldsymbol{x}_v\big) &= \boldsymbol{x}_{uu} \cdot \big((\boldsymbol{x}_u \cdot \boldsymbol{x}_v)\boldsymbol{x}_v - (\boldsymbol{x}_v \cdot \boldsymbol{x}_v)\boldsymbol{x}_u\big) = F\boldsymbol{x}_{uu} \cdot \boldsymbol{x}_v - G\boldsymbol{x}_{uu} \cdot \boldsymbol{x}_u \\
&= F\Big(F_u - \frac{1}{2}E_v\Big) - \frac{1}{2}GE_u \\
&= -(EG - F^2)\Gamma^1_{11} \\
\boldsymbol{x}_{uv} \cdot \big((\boldsymbol{x}_u \times \boldsymbol{x}_v) \times \boldsymbol{x}_u\big) &= \boldsymbol{x}_{uv} \cdot \big((\boldsymbol{x}_u \cdot \boldsymbol{x}_u)\boldsymbol{x}_v - (\boldsymbol{x}_v \cdot \boldsymbol{x}_u)\boldsymbol{x}_u\big) = E\boldsymbol{x}_{uv} \cdot \boldsymbol{x}_v - F\boldsymbol{x}_{uv} \cdot \boldsymbol{x}_u \\
&= \frac{1}{2}EG_u - \frac{1}{2}E_vF \\
&= (EG - F^2)\Gamma^2_{12} \\
\boldsymbol{x}_{uv} \cdot \big((\boldsymbol{x}_u \times \boldsymbol{x}_v) \times \boldsymbol{x}_v\big) &= \boldsymbol{x}_{uv} \cdot \big((\boldsymbol{x}_u \cdot \boldsymbol{x}_v)\boldsymbol{x}_v - (\boldsymbol{x}_v \cdot \boldsymbol{x}_v)\boldsymbol{x}_u\big) = F\boldsymbol{x}_{uv} \cdot \boldsymbol{x}_v - G\boldsymbol{x}_{uv} \cdot \boldsymbol{x}_u \\
&= \frac{1}{2}FG_u - \frac{1}{2}E_vG \\
&= -(EG - F^2)\Gamma^1_{12} \\
\boldsymbol{x}_{vv} \cdot \big((\boldsymbol{x}_u \times \boldsymbol{x}_v) \times \boldsymbol{x}_u\big) &= \boldsymbol{x}_{vv} \cdot \big((\boldsymbol{x}_u \cdot \boldsymbol{x}_u)\boldsymbol{x}_v - (\boldsymbol{x}_v \cdot \boldsymbol{x}_u)\boldsymbol{x}_u\big) = E\boldsymbol{x}_{vv} \cdot \boldsymbol{x}_v - F\boldsymbol{x}_{vv} \cdot \boldsymbol{x}_u \\
&= \frac{1}{2}EG_v - F\Big(F_v - \frac{1}{2}G_u\Big) \\
&= (EG - F^2)\Gamma^2_{22} \\
\boldsymbol{x}_{vv} \cdot \big((\boldsymbol{x}_u \times \boldsymbol{x}_v) \times \boldsymbol{x}_v\big) &= \boldsymbol{x}_{vv} \cdot \big((\boldsymbol{x}_u \cdot \boldsymbol{x}_v)\boldsymbol{x}_v - (\boldsymbol{x}_v \cdot \boldsymbol{x}_v)\boldsymbol{x}_u\big) = F\boldsymbol{x}_{vv} \cdot \boldsymbol{x}_v - G\boldsymbol{x}_{vv} \cdot \boldsymbol{x}_u \\
&= \frac{1}{2}FG_v - G\Big(F_v - \frac{1}{2}G_u\Big) \\
&= -(EG - F^2)\Gamma^1_{22} \\
\boldsymbol{x}_v \cdot \big((\boldsymbol{x}_u \times \boldsymbol{x}_v) \times \boldsymbol{x}_u\big) &= \boldsymbol{x}_v \cdot \big((\boldsymbol{x}_u \cdot \boldsymbol{x}_u)\boldsymbol{x}_v - (\boldsymbol{x}_v \cdot \boldsymbol{x}_u)\boldsymbol{x}_u\big) = EG - F^2.
\end{aligned}$$

Alltogether, we have

$$\begin{aligned}\kappa_{\mathrm{g}} = \frac{1}{\sqrt{EG-F^2}}\Big(&(u'v''-u''v')(EG-F^2)\\ &+(u')^3\Big(E\Big(F_u-\frac{1}{2}E_v\Big)-\frac{1}{2}FE_u\Big)+(u')^2v'\Big(F\Big(F_u-\frac{1}{2}E_v\Big)-\frac{1}{2}GE_u\Big)\\ &+2(u')^2v'\Big(\frac{1}{2}EG_u-\frac{1}{2}E_vF\Big)+2u'(v')^2\Big(\frac{1}{2}FG_u-\frac{1}{2}E_vG\Big)\\ &+u'(v')^2\Big(\frac{1}{2}EG_v-F\Big(F_v-\frac{1}{2}G_u\Big)\Big)+(v')^3\Big(\frac{1}{2}FG_v-G\Big(F_v-\frac{1}{2}G_u\Big)\Big)\Big)\\ = \sqrt{EG-F^2}\big(&\Gamma_{11}^2u'^3-\Gamma_{22}^1v'^3+(2\Gamma_{12}^2-\Gamma_{11}^1)u'^2v'-(2\Gamma_{12}^1-\Gamma_{22}^2)u'v'^2-u''v'+v''u'\big)\end{aligned}$$

In particular, for an arbitrarily parametrized curve the geodesic curvatire can be computed as

$$\kappa_{\mathrm{g}} = \frac{\sqrt{EG-F^2}\big(\Gamma_{11}^2u'^3-\Gamma_{22}^1v'^3+(2\Gamma_{12}^2-\Gamma_{11}^1)u'^2v'-(2\Gamma_{12}^1-\Gamma_{22}^2)u'v'^2-u''v'+v''u'\big)}{(Eu'^2+2Fu'v'+Gv'^2)^{3/2}}$$

(cf. Exercise 6.4).

7.3. Show that a curve of constant geodesic curvature c on the unit sphere $S^2(1)$ in $\mathbb{R}^3$ is a planar circle of length $2\pi(1+c^2)^{-1/2}$.

Hint: If $\boldsymbol{\alpha}$ is a curve of constant geodesic curvature c show that the vector $\boldsymbol{e}(s) = \boldsymbol{\alpha}(s)\times\boldsymbol{\alpha}'(s)+c\boldsymbol{\alpha}(s)$ does not depend on s, where $(\cdot)'$ denotes differentiation with respect to arc length).

Solution:

On the unit sphere we have $\boldsymbol{N}(\boldsymbol{\alpha}(s)) = \boldsymbol{\alpha}(s)$. Therefore,

$$\begin{aligned}\boldsymbol{e}(s) &= \boldsymbol{\alpha}(s)\times\boldsymbol{\alpha}'(s)+c\boldsymbol{\alpha}(s),\\ \boldsymbol{e}'(s) &= \underbrace{\boldsymbol{\alpha}'(s)\times\boldsymbol{\alpha}'(s)}_{=\boldsymbol{0}}+\underbrace{\boldsymbol{\alpha}(s)}_{=\boldsymbol{N}(\boldsymbol{\alpha}(s))}\times\boldsymbol{\alpha}''(s)+c\boldsymbol{\alpha}'(s),\\ &= \boldsymbol{N}(\boldsymbol{\alpha}(s))\times\Big(\kappa_{\mathrm{n}}\boldsymbol{N}(\boldsymbol{\alpha}(s))+\underbrace{\kappa_{\mathrm{g}}}_{=c}\big(\boldsymbol{N}(\boldsymbol{\alpha}(s))\times\boldsymbol{\alpha}'(s)\big)\Big)+c\boldsymbol{\alpha}'(s),\\ &= c\boldsymbol{N}(\boldsymbol{\alpha}(s))\times\big(\boldsymbol{N}(\boldsymbol{\alpha}(s))\times\boldsymbol{\alpha}'(s)\big)+c\boldsymbol{\alpha}'(s).\end{aligned}$$

Now, note that $\boldsymbol{a} := \boldsymbol{\alpha}'(s)$ and $\boldsymbol{b} := \boldsymbol{N}(\boldsymbol{\alpha}(s))$ are orthonormal vectors, therefore $\boldsymbol{c} := \boldsymbol{a}\times\boldsymbol{b}$ is also a unit vector orthogonal to $\boldsymbol{a}$ and $\boldsymbol{b}$. In particular, $\{\boldsymbol{a},\boldsymbol{b},\boldsymbol{c}\}$ is a positively oriented orthonormal basis of $\mathbb{R}^3$. For such a basis, we have $\boldsymbol{b}\times(\boldsymbol{b}\times\boldsymbol{a}) = -\boldsymbol{a}$, and hence $\boldsymbol{e}'(s) = \boldsymbol{0}$, so $\boldsymbol{e}(s) = \boldsymbol{e}$ is a constant vector.

We will now show that $\boldsymbol{\alpha}(s)$ lies in a plane: We have

$$\boldsymbol{\alpha}(s)\cdot\boldsymbol{e} = \boldsymbol{\alpha}(s)\cdot\big(\boldsymbol{\alpha}(s)\times\boldsymbol{\alpha}'(s)+c\boldsymbol{\alpha}(s)\big) = c\boldsymbol{\alpha}(s)\cdot\boldsymbol{\alpha}(s) = c$$

for all $s\in\mathbb{R}$ as $\boldsymbol{\alpha}(s)\in S^2(1)$. But this means that $\boldsymbol{\alpha}(s)$ makes a constant angle with $\boldsymbol{e}$ and thus lies in a plane at distance $c/\|\boldsymbol{e}(s)\|$ from the origin. Since $\|\boldsymbol{e}(s)\| = \sqrt{1+c^2}$ (by Pythagoras' theorem: $\{\boldsymbol{\alpha}(s)\times\boldsymbol{\alpha}'(s),\boldsymbol{\alpha}'(s)\}$ are orthonormal), the radius of the circle (the intersection of the plane with the unit sphere) is $r = \sqrt{1-c^2/(1+c^2)} = 1/\sqrt{1+c^2}$. Hence, the circle has circumference $2\pi r = 2\pi/\sqrt{1+c^2}$.

7.4. ($\star$) Let S be a surface in $\mathbb{R}^3$ and suppose that Π is a plane which intersects S orthogonally along a regular curve $\boldsymbol{\gamma}$. If $\boldsymbol{\alpha}(s)$ is a parametrization of $\boldsymbol{\gamma}$ such that $\|\boldsymbol{\alpha}'(s)\|$ is constant, show that $\boldsymbol{\alpha}$ is a geodesic of S.

Solution:

By construction, we have that $\boldsymbol{\alpha}'(s)$ and the normal $\boldsymbol{N}(\boldsymbol{\alpha}(s))$ are parallel to Π for all s. Let $\boldsymbol{e}$ be a non-zero vector normal to Π, then $\boldsymbol{N}(\boldsymbol{\alpha}(s))\times\boldsymbol{\alpha}'(s)$ is parallel to $\boldsymbol{e}$. From $\boldsymbol{\alpha}'(s)\cdot\boldsymbol{e} = 0$ (again, $\boldsymbol{\alpha}'(s)$

is parallel to Π) we deduce that (after taking the derivative) $\boldsymbol{\alpha}''(s)\cdot\boldsymbol{e}=0$ (as $\boldsymbol{e}$ is independent of s), so we see that

$$\kappa_{\mathrm{g}}=\frac{1}{\|\boldsymbol{\alpha}'(s)\|^3}\big(\boldsymbol{\alpha}'(s)\times\boldsymbol{\alpha}''(s)\big)\cdot\boldsymbol{N}(\boldsymbol{\alpha}(s))=\frac{1}{\|\boldsymbol{\alpha}'(s)\|^3}\big(\boldsymbol{N}(\boldsymbol{\alpha}(s))\times\boldsymbol{\alpha}'(s)\big)\cdot\boldsymbol{\alpha}''(s)=0.$$

Therefore, since the curve is also parametrized proportionally to arc length, it is a geodesic.

7.5. (a) Show that any constant speed curve on a surface S in $\mathbb{R}^3$ which is a curve of intersection of S with a plane of reflectional symmetry of S is a geodesic.

(b) Show that the curves of intersection of the coordinate planes in $\mathbb{R}^3$ with the surface S defined by the equation $x^4+y^6+z^8=1$ are geodesics.

Solution:

(a) A plane of reflection leaving a surface invariant intersects the surface orthogonally (prove this!). Therefore, the result follows immediately from the previous exercise.

(b) Note that the reflections $(x,y,z)\mapsto(-x,y,z)$, $(x,y,z)\mapsto(x,-y,z)$ and $(x,y,z)\mapsto(x,-y,z)$ all leave the surface given by $x^4+y^6+z^8=1$ invariant. Since these reflections are reflections along the coordinate planes, the result follows.

7.6. Let $\boldsymbol{\alpha}$ be a regular curve on a surface S in $\mathbb{R}^3$.

(a) If $\boldsymbol{\alpha}$ is both a line of curvature and a geodesic, show that $\boldsymbol{\alpha}$ is a planar curve.
Hint: Show that $\boldsymbol{N}\times\boldsymbol{\alpha}'$ is constant along $\boldsymbol{\alpha}$.

(b) If $\boldsymbol{\alpha}$ is both a geodesic and a planar curve with nowhere vanishing curvature show that $\boldsymbol{\alpha}$ is a line of curvature.

Solution:

(a) Denote

$$\boldsymbol{e}(s):=(\boldsymbol{N}\circ\boldsymbol{\alpha})(s)\times\boldsymbol{\alpha}'(s)$$

so that its derivative is

$$\begin{aligned}\boldsymbol{e}'(s)&=(\boldsymbol{N}\circ\boldsymbol{\alpha})'(s)\times\boldsymbol{\alpha}'(s)+(\boldsymbol{N}\circ\boldsymbol{\alpha})(s)\times\boldsymbol{\alpha}''(s)\\&=(d_{\boldsymbol{\alpha}(s)}\boldsymbol{N})(\boldsymbol{\alpha}'(s))\times\boldsymbol{\alpha}'(s)+\boldsymbol{N}(\boldsymbol{\alpha}(s))\times\boldsymbol{\alpha}''(s)\end{aligned}$$

Now, since $\boldsymbol{\alpha}$ is a line of curvature, $d_{\boldsymbol{\alpha}(s)}\boldsymbol{N}(\boldsymbol{\alpha}'(s))$ is a multiple of $\boldsymbol{\alpha}'(s)$, hence the first vector product vanishes (as $\boldsymbol{a}\times\boldsymbol{a}=\boldsymbol{0}$), and for the second term, note that as $\boldsymbol{\alpha}$ is a geodesic, $\boldsymbol{\alpha}''(s)$ is a multiple of $\boldsymbol{N}(\boldsymbol{\alpha}(s))$, and hence this vector product also vanishes. Alltogether we have shown $\boldsymbol{e}'(s)=0$ for all s, say

$$\boldsymbol{e}(s)=\boldsymbol{e}_0$$

for some vector $\boldsymbol{e}_0\in\mathbb{R}^3$. Note that $\boldsymbol{e}_0\neq\boldsymbol{0}$, because $\boldsymbol{\alpha}'$ and $\boldsymbol{\alpha}''$ are orthogonal (here we use the constant speed property of a geodesic), so $\boldsymbol{\alpha}'$ and $\boldsymbol{N}(\boldsymbol{\alpha}(s))$ are also orthogonal (and of course non-zero).

Let us now show that $\boldsymbol{\alpha}(s)$ lies in a plane normal to $\boldsymbol{e}_0$, i.e., $\boldsymbol{\alpha}(s)\cdot\boldsymbol{e}_0=\text{const}$, or equivalently, $\boldsymbol{\alpha}'(s)\cdot\boldsymbol{e}_0=0$. Indeed,

$$\boldsymbol{\alpha}'\cdot\boldsymbol{e}_0=\boldsymbol{\alpha}'\cdot\big((\boldsymbol{N}\circ\boldsymbol{\alpha})\times\boldsymbol{\alpha}'\big)=(\boldsymbol{N}\circ\boldsymbol{\alpha})\cdot\big(\boldsymbol{\alpha}'\times\boldsymbol{\alpha}'\big)=0.$$

In particular, $\boldsymbol{\alpha}(s)$ lies in a plane for all s.

(b) If $\boldsymbol{\alpha}$ is a geodesic, then $\boldsymbol{\alpha}''=\kappa_{\mathrm{n}}\boldsymbol{N}$. Moreover, there exist $\boldsymbol{e}_0\in\mathbb{R}^3$ such that $\boldsymbol{\alpha}(s)\cdot\boldsymbol{e}_0=\text{const}$ for all s (as $\boldsymbol{\alpha}$ lies in a plane), hence taking the derivatives give $\boldsymbol{\alpha}'\cdot\boldsymbol{e}_0=0$ and $\boldsymbol{\alpha}''\cdot\boldsymbol{e}_0=0$. Using the fact that $\kappa_{\mathrm{n}}(s)\neq0$ for all s we conclude that $\boldsymbol{e}_0$ is orthogonal to $\boldsymbol{N}(\boldsymbol{\alpha}(s))$ and $\boldsymbol{\alpha}'(s)$ for all s. Taking the derivative of $\boldsymbol{N}(\boldsymbol{\alpha}(s))\cdot\boldsymbol{e}_0=0$ gives

$$d_{\boldsymbol{\alpha}(s)}\boldsymbol{N}(\boldsymbol{\alpha}'(s))\cdot\boldsymbol{e}_0=0$$

for all s, and from the fact that $\boldsymbol{N}$ is a unit vector, we also obtain that $d_{\boldsymbol{\alpha}(s)}\boldsymbol{N}(\boldsymbol{\alpha}'(s))$ is orthognal to $\boldsymbol{N}(\boldsymbol{\alpha}(s))$. In particular we have shown that $d_{\boldsymbol{\alpha}(s)}\boldsymbol{N}(\boldsymbol{\alpha}'(s))$ and $\boldsymbol{\alpha}'(s)$ both are orthogonal to $\boldsymbol{e}_0$ and $d_{\boldsymbol{\alpha}(s)}\boldsymbol{N}(\boldsymbol{\alpha}'(s))$, hence there must be a scalar $\lambda(s)\in\mathbb{R}$ such that $d_{\boldsymbol{\alpha}(s)}\boldsymbol{N}(\boldsymbol{\alpha}'(s))=\lambda(s)\boldsymbol{\alpha}'(s)$, i.e., $\boldsymbol{\alpha}$ is a line of curvature.