Durham University
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Differential Geometry III, Solutions 8 (Week 18)

Geodesics – 2.

8.1. Find all the geodesics on the flat torus $S^1(1) \times S^1(1) \subset \mathbb{R}^4$, where $S^1(1)$ is the circle of radius 1 in $\mathbb{R}^2$ centered at the origin. Prove that there are infinitely many both closed and non-closed geodesics through the point $(1, 0, 1, 0) \in S^1(1) \times S^1(1)$.

Solution:

The plane $\mathbb{R}^2$ and the flat torus $T = S^1(1) \times S^1(1)$ are locally isometric via

$$f(u, v) = (\cos u, \sin u, \cos v, \sin v)$$

(as it can be easily seen from $f_u \cdot f_u = 1$, $f_u \cdot f_v = 0$ and $f_v \cdot f_v = 0$, and the fact that $E = G = 1$, $F = 0$ are also the coefficients of the first fundamental form of the plane). Local isometries preserve geodesics, hence images of lines under f are geodesics of T: examples through $(1, 0, 1, 0)$ are

$$\boldsymbol{\alpha}_r : \mathbb{R} \longrightarrow T, \qquad \boldsymbol{\alpha}_{p,q}(s) = (\cos(ps), \sin(ps), \cos(qs), \sin(qs)).$$

for some $p, q \in \mathbb{R}$ such that $p^2 + q^2 = 1$ (these are images of the lines $s \mapsto (ps, qs)$ in the plane, having unit speed). Note that $\boldsymbol{\alpha}_r(0) = (1, 0, 1, 0)$. Moreover, if $r = p/q$ is rational (w.l.o.g., p, q both rational, say, $p = a/c$ and $q = b/c$, $a, b \in \mathbb{Z}$, $c \in \mathbb{N}$), then $\boldsymbol{\alpha}_{p,q}(s + 2\pi c) = \boldsymbol{\alpha}_{p,q}(s)$ and hence $\boldsymbol{\alpha}_{p,q}$ is a closed geodesic on T. Obviously, there are infinitely many such parameters p and q.

If p/q is irrational, then $\boldsymbol{\alpha}_{p,q}(s_1) = \boldsymbol{\alpha}_{p,q}(s_2)$ implies $p(s_1 - s_2), q(s_1 - s_2) \in 2\pi\mathbb{Z}$, i.e.,

$$2\pi(s_1 - s_2) \in (p^{-1}\mathbb{Z}) \cap (q^{-1}\mathbb{Z}).$$

But since p/q is irrational, the latter set only contains $\{0\}$, and hence $s_1 = s_2$, i.e., the curve $\boldsymbol{\alpha}_{p,q}$ is injective, i.e., the line $\mathbb{R}$ is embedded injectively into T. Again, there are infinitely many such parameters p and q.

8.2. Let $\mathbb{H}$ be the hyperbolic plane, i.e. the surface $\mathbb{R} \times (0, \infty)$ with coefficients of the first fundamental form $E(u, v) = G(u, v) = 1/v^2$ and $F(u, v) = 0$. Show that the geodesics in $\mathbb{H}$ are the intersections of $\mathbb{H}$ with the lines and circles in $\mathbb{R}^2$ which meet the u-axis orthogonally.

Hint: After obtaining the differential equations you may not try to solve them but, instead, just check that the curves above are indeed geodesics, and then prove that there are no others.

Solution:

The equation of a geodesic in a local parametrization is

$$u''E + \frac{1}{2}(u')^2 E_u + u'v'E_v + (v')^2\Big(F_v - \frac{1}{2}G_u\Big) + v''F = 0,$$
$$v''G + \frac{1}{2}(v')^2 G_v + u'v'G_u + (u')^2\Big(F_u - \frac{1}{2}E_v\Big) + u''F = 0,$$

which reduces here to

$$\begin{aligned}\frac{1}{v^2}u'' - 2\frac{1}{v^3}u'v' &= 0,\\ \frac{1}{v^2}v'' - \frac{1}{v^3}(v')^2 + \frac{1}{v^3}(u')^2 &= 0,\end{aligned}$$

which is equivalent to

$$\left(\frac{u'}{v^2}\right)' = 0, \quad \text{and} \quad v'' + \frac{u'^2 - v'^2}{v} = 0.$$

These are equivalent to

$$u' = cv^2 \quad \text{and} \quad v'' + \frac{u'^2 - v'^2}{v} = 0$$

for some constant $c \in \mathbb{R}$.

Consider vertical lines first, i.e. $u = u_0$. Then the first equation clearly holds for $c = 0$, and the second reduces to $v'' = \frac{v'^2}{v}$, which also holds if we parametrize a vertical line by $v(s) = ke^s$ for any positive k.

Consider now semicircles orthogonal to the real axis, each of these can be parametrized by

$$\boldsymbol{\alpha}(s) = (u(s), v(s)) = (u_0 + a\cos f(s), a\sin f(s))$$

for some $u_0 \in \mathbb{R}$, $a \in \mathbb{R}_{>0}$ and a smooth monotone function f. The first equation then becomes

$$f'(s) = -ca\sin f(s),$$

so assume the function f satisfies this. We need to verify that the second equation is then also fulfilled. In view of the relation above, we have

$$\begin{aligned} v'(s) =& af'(s)\cos f(s) = & -ca^2\sin f(s)\cos f(s) = \frac{-ca^2}{2}\sin 2f(s),\\ v''(s) =& \frac{-ca^2}{2}2f'(s)\cos 2f(s) = & c^2a^3\sin f(s)\cos 2f(s),\\ u'(s) =& ca^2\sin^2 f(s). & \end{aligned}$$

Therefore,

$$v'' + \frac{u'^2 - v'^2}{v} = c^2a^3\sin f(s)\cos 2f(s) + \frac{c^2a^4\sin^4 f(s) - c^2a^4\sin^2 f(s)\cos^2 f(s)}{a\sin f(s)} = c^2a^3\sin f(s)(\cos 2f(s) + (\mathrm{si}$$

so the second equation also holds.

Finally, for a given point $p \in \mathbb{H}$ and a tangent vector $w \in T_p\mathbb{H}$ there exists a unique circle (or line) through p and tangent to w intersecting the real axis orthogonally. By the uniqueness theorem, this implies that there are no other geodesics except for the ones described above.

8.3. How many closed geodesics are there on the surface of revolution in $\mathbb{R}^3$ obtained by rotating the curve $z = 1/x^2$, $(x > 0)$ around the z-axis?

Solution:

Assume that $\boldsymbol{\alpha}(s) = \boldsymbol{x}(u(s), v(s))$ is a closed geodesic, where

$$\boldsymbol{x}(u, v) = f(v)\cos u, f(v)\sin u, \frac{1}{f^2(v)},$$

where f is monotonic and the curve $f(v), 1/f^(v)$ has unit speed. The Clairaut relation then says that

$$f(v(s))\cos\Theta(s) = \text{const},$$

where $\Theta(s)$ is the angle formed by $\boldsymbol{\alpha}'(s)$ and the parallel through $\boldsymbol{\alpha}(s)$. Let $z_{\min}$ and $z_{\max}$ be the minimal and maximal values of z along $\boldsymbol{\alpha}$, denote $z_{\min} = 1/f(v(s_{\min}))$ and $z_{\max} = 1/f(v(s_{\max}))$, these defines the values of s uniquely since f is monotonic. Then $\Theta(s_{\min}) = \Theta(s_{\max}) = 0$, which implies that $f(v(s_{\min})) = f(v(s_{\max}))$, so $z_{\min} = z_{\max}$, i.e. $\boldsymbol{\alpha}$ must be a parallel. However, it is easy to see that no parallel is a geodesic (as $f'(v)$ never vanishes). This proves that the surface has no closed geodesics.

8.4. ($\star$) Let S be the cone obtained by rotating the line $z = \beta x$ ($z > 0$) around the z-axis, where β is a positive constant. Let $\boldsymbol{\alpha}(s) = (x(s), y(s), z(s))$ be a geodesic on S intersecting the parallel $z = 1$ at an angle ϑ_0. Find the smallest value of $z(s)$. Investigate whether $\boldsymbol{\alpha}$ has self-intersections.

Solution:

Parametrize the generating curve by $(v, 0, \beta v)$. Then the Clairaut equation reduces to

$$v(s)\cos\vartheta(s) = \text{const},$$

where $\vartheta(s)$ is the angle formed by $\boldsymbol{\alpha}$ with the parallel at $\boldsymbol{\alpha}(s)$.The constant here is the value of $v(s)\cos\vartheta$ at $z = 1$, i.e. at $v = 1/\beta$. Thus, we have an equation

$$v(s)\cos\vartheta(s) = \frac{\cos\vartheta_0}{\beta}.$$

By symmetry, at the point $\boldsymbol{\alpha}(s_0)$ of $\boldsymbol{\alpha}$ closest to the origin the angle $\vartheta(s_0)$ is equal to zero, so $v(s_0) = \cos\vartheta_0/\beta$. Therefore,

$$z(s_0) = \beta v(s_0) = \cos(\vartheta_0),$$

so it is independent of β! Note that if $\vartheta_0 = \pi/2$, then the distance is 0 which means that the geodesic goes through the apex.

Alternatively, we could use the fact the geodesics on a cone are just images of lines under the local isometry between the plane and a cone. In particular, by considering the preimage of the cone under an isometry in $\mathbb{R}^2$, one can easily see that $\boldsymbol{\alpha}$ is self-intersecting if and only if the total angle of the cone in the apex is strictly less that π. By Pythagoras' Theorem, the latter is equivalent to $2/\sqrt{1+\beta^2} < 1$, which is the same as $\beta > \sqrt{3}$ or $\arctan\beta > \pi/3$.

8.5. Let $\boldsymbol{\alpha}\colon I \longrightarrow \mathbb{R}^3$ be a curve parametrized by arc length with everywhere non-zero curvature, and let $\boldsymbol{b}(s)$ be a vector such that the map

$$\boldsymbol{x}(s, v) = \boldsymbol{\alpha}(s) + v\boldsymbol{b}(s), \qquad s \in I, v \in (-\epsilon, \epsilon),$$

is a parametrization of a regular surface S for some $\varepsilon > 0$ (S is a ruled surface — you don't have to show that the surface is regular).

(a) Is the curve $\boldsymbol{\beta}\colon (-\varepsilon, \varepsilon) \longrightarrow S$ given by $\boldsymbol{\beta}(v) = \boldsymbol{x}(s_0, v)$ for some $s_0 \in I$ a geodesic? Justify your answer.

(b) Assume now that $\boldsymbol{b}(s)$ is the binormal of the space curve $\boldsymbol{\alpha}$ at $\boldsymbol{\alpha}(s)$. Prove that $\boldsymbol{\alpha}$ is a geodesic on S (i.e., show that the *generating* curve is a geodesic on the ruled surface generated by a curve and its binormal.)

Solution:

(a) Any line in a surface is a geodesic (as in its standard parametrisation, $\boldsymbol{\alpha}(s) = p + s\boldsymbol{v}$ has derivatives $\boldsymbol{\alpha}'(s) = \boldsymbol{v}$ and $\boldsymbol{\alpha}''(s) = \boldsymbol{0}$, hence $\kappa_{\mathrm{g}} = 0$.

(b) The normal $\boldsymbol{N}_{\boldsymbol{\alpha}}$ and binormal $\boldsymbol{b}$ of the curve $\boldsymbol{\alpha}$ are given by

$$\boldsymbol{N}_{\boldsymbol{\alpha}}(s) = \frac{1}{\|\boldsymbol{\alpha}''(s)\|}\boldsymbol{\alpha}''(s) \qquad \text{and} \qquad \boldsymbol{b}(s) = \boldsymbol{\alpha}'(s) \times \boldsymbol{N}_{\boldsymbol{\alpha}}(s)$$

(assuming that $\boldsymbol{\alpha}''(s) \neq 0$, see Section 4 of the notes of the first term). The two tangent vectors of the ruled surface are $\boldsymbol{x}_s = \boldsymbol{\alpha}'$ and $\boldsymbol{x}_v = \boldsymbol{b}$, hence the normal vector $\boldsymbol{N}$ of the surface is

$$\boldsymbol{N} = \frac{1}{\|\boldsymbol{\alpha}' \times \boldsymbol{b}\|}\boldsymbol{\alpha}' \times \boldsymbol{b},$$

and hence, the vector $\boldsymbol{N} \times \boldsymbol{\alpha}'$ is proportional to $(\boldsymbol{\alpha}' \times \boldsymbol{b}) \times \boldsymbol{\alpha}'$ and therefore proportional to $\boldsymbol{b}$ (since $\boldsymbol{\alpha}'$ and $\boldsymbol{b}$ are orthonormal). Now, $\boldsymbol{b}$ is, by definition, orthogonal to $\boldsymbol{\alpha}''$ and hence $\kappa_{\mathrm{g}} = \boldsymbol{\alpha}'' \cdot (\boldsymbol{N} \times \boldsymbol{\alpha}') = 0$.

Alternative solution: You can also verify that $\boldsymbol{\alpha}''(s)$ is orthogonal to $T_{\boldsymbol{\alpha}(s)}S$ for all s by checking

$$\boldsymbol{\alpha}'' \cdot \boldsymbol{x}_s = \boldsymbol{\alpha}'' \cdot \boldsymbol{\alpha}' \overset{!}{=} 0 \qquad \text{and} \qquad \boldsymbol{\alpha}'' \cdot \boldsymbol{x}_v = \boldsymbol{\alpha}'' \cdot \boldsymbol{b} \overset{!}{=} 0.$$

Now, $\boldsymbol{\alpha}'' \cdot \boldsymbol{\alpha}' = 0$ as $\|\boldsymbol{\alpha}'\|^2 = 1$, which implies $2\boldsymbol{\alpha}'' \cdot \boldsymbol{\alpha}' = 0$. Moreover, $\boldsymbol{b}$ is by definition orthogonal to $\boldsymbol{\alpha}''$, and hence the second orthogonality condition is also satisfied.