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Differential Geometry III, Solutions 9 (Week 19)

Gauss–Bonnet Theorem

9.1. Show that the catenoid $x^2 + y^2 = \cosh^2 z$ has a unique closed geodesic.

Solution:

There is one obvious geodesic $\boldsymbol{\alpha}_0$: it is the intersection of the surface with the xy-plane. Indeed, note that this is a *parallel*, and a parallel at v_0 is a geodesic iff $f'(v_0) = 0$ (using the standard parametrization of a surface of revolution). Here, $f(v) = \cosh v$, and $v_0 = 0$, so that $f'(0) = \sinh 0 = 0$. Thus, we have found a closed geodesic.

Assume now that there is another closed geodesic. If it does not intersect $\boldsymbol{\alpha}_0$, then both enclose a region R homeomorphic to a cylinder, hence having Euler characteristic $\chi(R) = 0$.

The Gauss curvature is everywhere negative and you can find that

$$K = -\frac{1}{\cosh^2 v}.$$

Therefore, by the Gauss–Bonnet theorem we have

$$\int_R K \,\mathrm{d}A = 2\pi\chi(R) = 0,$$

a contradiction (note that there is no line integral term as the boundary of R consists of geodesics, and there is no angle term either, as both geodesics are closed).

If the second geodesic intersects the first one, they enclose a region R homeomorphic to a disc, hence $\chi(R) = 1$. There are now two vertices from the intersection of the two geodesics. Therefore, by the Gauss–Bonnet theorem we have

$$\int_R K \,\mathrm{d}A + \vartheta_1 + \vartheta_2 = 2\pi\chi(R) = 2\pi.$$

As $K < 0$, we must have $\vartheta_1 + \vartheta_2 > 2\pi$ which is impossible (both angles are smaller than π by definition).

9.2. Find the Gauss curvature K at all points of the surface given by $x^2 + y^2 = z$. Evaluate $\int_R K \,\mathrm{d}A$, where R is the region of the surface from $z = 0$ to $z = a^2$ ($a \in \mathbb{R}$).

Solution:

We can parametrize the surface as a surface of revolution (although it does not cover the bottom point $(0, 0, 0)$, but for integration, this point does not matter too much).

The surface can be parametrized by

$$\boldsymbol{x}(u,v) = (v\cos u, v\sin u, v^2)$$

for $u \in (0, 2\pi)$ (or $u \in (-\pi, \pi)$) and $v \in (0, \infty)$. Then by standard calculations, the Gauss curvature in this parametrization if given by

$$K = \frac{-g'(f''g' - f'g'')}{f(f'^2 + g'^2)^2} = \frac{-2v(-2)}{v(1+4v^2)^2} = \frac{4}{(1+4v^2)^2}$$

for $f(v) = v$, $g(v) = v^2$.

Moreover, the area element is $\sqrt{EG - F^2}\,\mathrm{d}u\,\mathrm{d}v$, i.e., here ($E = f^2 = v^2$, $F = 0$, $G = f'^2 + g'^2 = 1 + 4v^2$) we have

$$\mathrm{d}A = \sqrt{EG - F^2}\,\mathrm{d}u\,\mathrm{d}v = \sqrt{v^2(1+4v^2)}\,\mathrm{d}u\,\mathrm{d}v = v\sqrt{1+4v^2}\,\mathrm{d}u\,\mathrm{d}v$$

(as $v > 0$). Now, R is the region covered by the parameter domain $(u,v) \in (0, 2\pi) \times (0, a)$, and

$$\begin{aligned}
\int_R K\,\mathrm{d}A &= \int_0^a \int_0^{2\pi} K\sqrt{EG - F^2}\,\mathrm{d}u\,\mathrm{d}v \\
&= \int_0^a \int_0^{2\pi} \frac{4v}{(1+4v^2)^{3/2}}\,\mathrm{d}u\,\mathrm{d}v \\
&= 2\pi \int_0^a \frac{4v}{(1+4v^2)^{3/2}}\,\mathrm{d}v \\
&= \pi \int_1^{1+4a^2} \frac{1}{t^{3/2}}\,\mathrm{d}t \\
&= \pi\Big[-2t^{-1/2}\Big]_1^{1+4a^2} = 2\pi\left(1 - \frac{1}{\sqrt{1+4a^2}}\right)
\end{aligned}$$

substituting $1 + 4v^2 = t$ (i.e., $8v\,\mathrm{d}v = \mathrm{d}t$) in the fourth equation.

9.3. Verify the Gauss–Bonnet Theorem directly for the region R in the previous question.

Solution:

The Gauss–Bonnet theorem in the given situation is

$$\int_R K\,\mathrm{d}A + \int_{\partial R} \kappa_{\mathrm{g}}\,\mathrm{d}s = 2\pi\chi(R).$$

The boundary of R is the circle $\boldsymbol{\alpha}(s) = (a\cos(s/a), a\sin(s/a), a^2)$ (the factor $1/a$ guarantees that $\boldsymbol{\alpha}$ is parametrized by arc length).

We have

$$\boldsymbol{\alpha}'(s) = (-\sin(s/a), \cos(s/a), 0), \qquad \boldsymbol{\alpha}''(s) = (-a^{-1}\cos(s/a), -a^{-1}\sin(s/a), 0).$$

The normal of a surface given by the equation $h(x,y,z) = 0$ is given by normalizing ∇h: here we have $h(x,y,z) = z - x^2 - y^2$, hence $\nabla h = (-2x, -2y, 1)$, and the normal vector at $p = (x,y,z)$ is

$$\boldsymbol{N}(x,y,z) = \frac{1}{\sqrt{4x^2 + 4y^2 + 1}}(-2x, -2y, 1).$$

Note that we have chosen the normal vector to point *upwards* so that it is consistent with the direction of the curve (counterclockwise).

Hence, the geodesic curvature can be calculated by (note that $u = s/a$ and $v = a$ on $\boldsymbol{\alpha}$: $\boldsymbol{N}$ is evaluated at $\boldsymbol{\alpha}(s)$)

$$\kappa_g = \alpha'' \cdot (\boldsymbol{N} \times \alpha') = \begin{pmatrix} -a^{-1}\cos(s/a) \\ -a^{-1}\sin(s/a) \\ 0 \end{pmatrix} \cdot \left(\frac{1}{\sqrt{4a^2+1}} \begin{pmatrix} -2a\cos(s/a) \\ -2a\sin(s/a) \\ 1 \end{pmatrix} \times \begin{pmatrix} -\sin(s/a) \\ \cos(s/a) \\ 0 \end{pmatrix} \right)$$
$$= \frac{1}{\sqrt{4a^2+1}} \begin{pmatrix} -a^{-1}\cos(s/a) \\ -a^{-1}\sin(s/a) \\ 0 \end{pmatrix} \cdot \begin{pmatrix} -\cos(s/a) \\ -\sin(s/a) \\ -2a \end{pmatrix} = \frac{1}{a\sqrt{4a^2+1}}$$

(note that the sign of κ_g depends on the choice of the sign of $\boldsymbol{N}$, cf. the first problem from the last problems class!), so that the line integral over $\boldsymbol{\alpha}$ gives

$$\int_{\partial R} \kappa_g \, \mathrm{d}s = \frac{2\pi a}{a\sqrt{4a^2+1}} = \frac{2\pi}{\sqrt{4a^2+1}}$$

as the function under the intergral is constant. Using the calculations above, we obtain

$$\int_R K \, \mathrm{d}A + \int_{\partial R} \kappa_g \, \mathrm{d}s = 2\pi\left(1 - \frac{1}{\sqrt{1+4a^2}}\right) + \frac{2\pi}{\sqrt{4a^2+1}} = 2\pi$$

Finally, R is homeomorphic to a disc, so that its Euler characteristic is equal to 1, and hence we have verified the Gauss–Bonnet theorem in this case.

9.4. Let S be a connected compact orientable surface in $\mathbb{R}^3$ which is not homeomorphic to a sphere. Prove that S contains elliptic points, hyperbolic points and flat points.

Solution:

Since S is a compact surface not homeomorphic to a sphere, so it must have genus at least 1 and hence the Euler characteristic fulfils $\chi(S) \leq 0$. By the Gauss–Bonnet Theorem, we have

$$\int_S K \, \mathrm{d}A = 2\pi\chi(S) \leq 0$$

Moreover, we know that there is at least one point with $K(p) > 0$, i.e., an elliptic point (see Theorem 9.17). Since the integral is non-positive, there must be also hyperbolic points ($K(p) < 0$) and by continuity points with $K(p) = 0$.

9.5. Let S be a minimal surface homeomorphic to a plane. Show that two geodesics have at most one point of intersection.

Solution:

Since S is minimal, the mean curvature vanishes identically, which implies that $K \leq 0$. Further considerations are similar to Example 13.12 and the first exercise from this sheet.

Indeed, assume two geodesics intersect at more than one point, choose two consecutive points of intersection. Then the two segments of these geodesics form a region R with two exterior angles ϑ_1 and ϑ_2. By Gauss–Bonnet theorem we have

$$\underbrace{\int_R K \, \mathrm{d}A}_{\leq 0} + \underbrace{\int_{\partial R} \kappa_g \, \mathrm{d}s}_{=0} + \vartheta_1 + \vartheta_2 = 2\pi,$$

which cannot happen as ϑ_1 and ϑ_2 are less than π (note that in general the exterior angles may equal π, but in our case this would contradict the uniqueness of a geodesic with a given tangent vector).

Further problem

9.6. Let S be a surface of revolution with local parametrization

$$\mathbf{x}(u,v) = \big(f(v)\cos(u),\ f(v)\sin(u),\ v\big)$$

where $f(v)$ is a positive function of v. Suppose that S is minimal.

(a) Show that f satisfies:
$$\frac{f''(v)}{1+(f'(v))^2} = \frac{1}{f(v)}.$$

(b) Multiplying both sides of this equation by $2f'(v)$ and integrating, show that for some k:
$$1+(f'(v))^2 = k^2 f(v)^2$$

(c) Rearranging and integrating, show that for some c
$$f(v) = \frac{1}{k}\cosh(kv+c).$$

The conclusion is that S is (part of) a catenary.

Solution:

(a) Calculating the coefficients of the first and second fundamental forms, we find:

$$E = f^2(v), \qquad F = 0, \qquad G = (f'(v))^2 + 1$$
$$L = \frac{-f(v)}{\sqrt{(f'(v))^2+1}}, \qquad M = 0, \qquad N = \frac{f''(v)}{\sqrt{(f'(v)^2+1}}.$$

Therefore, the mean curvature is:

$$H = \frac{EN - 2FM + GL}{2(EG-F^2)} = \frac{-((f'(v))^2+1) + f(v)f''(v)}{2f(v)((f'(v))^2+1)^{3/2}}.$$

If $H = 0$ then we have
$$\frac{f''(v)}{1+(f'(v))^2} = \frac{1}{f(v)}$$
as claimed.

(b) Multiplying both sides by $2f'(v)$ gives:

$$\frac{2f''(v)f'(v)}{1+(f'(v))^2} = \frac{2f'(v)}{f(v)}.$$

Integrating gives

$$\log\big(1+(f'(v))^2\big) = 2\log(f(v)) + 2\log(k) = \log(k^2 f(v)^2)$$

where we write the constant of integration as $2\log(k)$. Exponentiating gives

$$1+(f'(v))^2 = k^2 f(v)^2$$

as claimed.

(c) Rearranging this gives

$$\frac{kf'(v)}{\sqrt{(kf(v))^2 - 1}} = k.$$

Integrating gives

$$\operatorname{arccosh}\big(kf(v)\big) = kv + c$$

where c is the constant of integration. In other words

$$f(v) = \frac{1}{k}\cosh(kv + c)$$

as claimed.