Evolute and involute

3.1. Let $\alpha$ denote the catenary from Exercise 2.1. Show that

(a) the involute of $\alpha$ starting from $(0, 1)$ is the tractrix from Exercise 1.6 (with $x$- and $y$-axes exchanged and different parametrization);

(b) the evolute of $\alpha$ is the curve given by

$$\beta(u) = (u - \sinh u \cosh u, 2 \cosh u)$$

(c) Find the singular points of $\beta$ and give a sketch of its trace.

Solution:

(a) The involute of $\alpha$ has parametrization

$$\gamma(u) = \alpha(u) - \ell(u)t(u)$$

Since $\alpha'(u) = (1, \sinh u)$, we have

$$\ell(u) = \int_0^u \|\alpha'(v)\| \, dv = \int_0^u \cosh v \, dv = \sinh u$$

and

$$t(u) = \frac{1}{\cosh u} (1, \sinh u),$$

so

$$\gamma(u) = \alpha(u) - \ell(u)t(u) = \left( u - \frac{\sinh u}{\cosh u}, \cosh u - \frac{\sinh^2 u}{\cosh u} \right) = \frac{1}{\cosh u} (u \cosh u - \sinh u, 1)$$

Exchanging coordinate axes, we obtain a curve parametrized by

$$\tilde{\gamma}(u) = \frac{1}{\cosh u} (1, u \cosh u - \sinh u)$$

The tractrix from Exercise 1.6 is completely characterized by its property (d). Computing the corresponding distance for the curve $\tilde{\gamma}(u)$ we see that its trace is also a tractrix.

(b) As we have already computed in Exercise 2.1 and in (a),

$$t(u) = \frac{1}{\cosh u} (1, \sinh u), \quad \kappa(u) = \frac{1}{\cosh^2 u}$$

In particular, $\kappa(u)$ is never zero, and

$$n(u) = \frac{1}{\cosh u} (-\sinh u, 1)$$

Now we can compute the evolute:

$$\epsilon(u) = \alpha(u) + \frac{1}{\kappa(u)}n(u) = (u - \sinh u \cosh u, 2 \cosh u)$$

as required.

(c) The singular points of $\epsilon$ correspond to the vertices of $\alpha$. We have

$$\kappa'(u) = \left( \frac{1}{\cosh^2 u} \right)' = -\frac{2 \sinh u}{\cosh^3 u},$$

so $\kappa'(u) = 0$ if and only if $u = 0$. The only singular point of $\epsilon$ is $(0, 2)$. 

3.2. (*Parallels. Let \( \alpha \) be a plane curve parametrized by arc length, and let \( d \) be a real number. The curve \( \beta(u) = \alpha(u) + d\mathbf{n}(u) \) is called the parallel to \( \alpha \) at distance \( d \).

(a) Show that \( \beta \) is a regular curve except for values of \( u \) for which \( d = 1/\kappa(u) \), where \( \kappa \) is the curvature of \( \alpha \).

(b) Show that the set of singular points of all the parallels (i.e., for all \( d \in \mathbb{R} \)) is the evolute of \( \alpha \).

Solution:

(a) Assume \( \kappa(u) = 0 \) or \( d\kappa(u) \neq 1 \). The latter is automatically satisfied if \( \kappa(u) = 0 \). So we just assume that \( d\kappa(u) \neq 1 \). We need to show that \( \beta'(u) \neq 0 \). Since \( \alpha \) is unit speed, we have

\[
\beta'(u) = t(u) + d\mathbf{n}'(u) = t(u) + d\mathbf{A}'(u) = \mathbf{A}(u)A^2t(u) = t(u) - d\kappa(u)t(u) = (1 - d\kappa(u))t(u),
\]

with \( A = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} \) and vectors \( t \) and \( \mathbf{n} \) are understood as columns. Note that \( \|t(u)\| = 1 \), i.e., \( t(u) \neq 0 \).

The initial assumption implies that \( (1 - d\kappa(u)) \neq 0 \) and, therefore, \( \beta'(u) \neq 0 \), i.e., \( \beta(u) \) is regular.

In the case \( \kappa(u) \neq 0 \) and \( d\kappa(u) = 1 \), i.e., \( d = 1/\kappa(u) \), we obviously have \( \beta'(u) = 0 \), i.e., \( \beta(u) \) is singular.

(b) The evolute is only defined in the case that we have \( \kappa(u) \neq 0 \) for all \( u \). So we assume this. We have seen in (a) that the singular points of the parallels are precisely those \( \beta(u) \) for which we have \( d\kappa(u) = 1 \), i.e., \( d = 1/\kappa(u) \). This means that the set of singular points of all parallels is

\[
\{\alpha(u) + d\mathbf{n}(u) \mid u \in I, \ d = 1/\kappa(u)\} = \{\alpha(u) + \frac{1}{\kappa(u)}\mathbf{n}(u) \mid u \in I\}
\]

which is precisely the parametrization of the evolute of \( \alpha \).

3.3. Let \( \alpha(u) : I \to \mathbb{R}^2 \) be a smooth regular curve. Suppose there exists \( u_0 \in I \) such that the distance \( \|\alpha(u_0)\| \) from the origin to the trace of \( \alpha \) is maximal at \( u_0 \). Show that the curvature \( \kappa(u_0) \) of \( \alpha \) at \( u_0 \) satisfies

\[
|\kappa(u_0)| \geq 1/\|\alpha(u_0)\|
\]

Solution:

Note first that the both sides of the inequality we want to prove do not depend on the parametrization, so we may assume without loss of generality that \( \alpha \) is parametrized by arc length.

Consider the function \( f(u) = \|\alpha(u)\|^2 \). Since \( f(u) \) has a maximum at \( u_0 \), the first derivative of \( f(u) \) at \( u_0 \) vanishes (cf. Exercise 1.4(b)), and the second derivative is non-positive. Thus, we have

\[
0 \geq f''(u_0) = (\alpha(u) \cdot \alpha(u))''|_{u_0} = (2\alpha'(u) \cdot \alpha(u))'|_{u_0} = \alpha''(u_0) \cdot \alpha(u_0) + 2\|\alpha'(u_0)\|^2 = \alpha''(u_0) \cdot \alpha(u_0) + 2
\]

To satisfy the inequality above, we must have \( \alpha''(u_0) \cdot \alpha(u_0) \leq -1 \), which implies \( |\alpha''(u_0) \cdot \alpha(u_0)| \geq 1 \), and therefore

\[
|\kappa(u_0)| = \|\alpha''(u_0)\| \geq 1/\|\alpha(u_0)\|
\]

3.4. Contact with circles. The points \((x, y) \in \mathbb{R}^2\) of a circle are given as solutions of the equation \( C(x, y) = 0 \) where

\[
C(x, y) = (x - a)^2 + (y - b)^2 - \lambda
\]

Let \( \alpha = (x(u), y(u)) \) be a plane curve. Suppose that the point \( \alpha(u_0) \) is also on some circle defined by \( C(x, y) \). Then \( C \) vanishes at \((x(u_0), y(u_0))\) and the equation \( g(u) = 0 \) with

\[
g(u) = C(x(u), y(u)) = (x(u) - a)^2 + (y(u) - b)^2 - \lambda
\]

has a solution at \( u_0 \). If \( u_0 \) is a multiple solution of the equation, with \( g^{(i)}(u_0) = 0 \) for \( i = 1, \ldots, k-1 \) but \( g^{(k)}(u_0) \neq 0 \), we say that the curve \( \alpha \) and the circle have \( k \)-point contact at \( \alpha(u_0) \).
(a) Let a circle be tangent to \( \alpha \) at \( \alpha(u_0) \). Show that \( \alpha \) and the circle have at least 2-point contact at \( \alpha(u_0) \).

(b) Suppose that \( \kappa(u_0) \neq 0 \). Show that \( \alpha \) and the circle have at least 3-point contact at \( \alpha(u_0) \) if and only if the center of the circle is the center of curvature of \( \alpha \) at \( \alpha(u_0) \).

(c) Show that \( \alpha \) and the circle have at least 4-point contact if and only if the center of the circle is the center of curvature of \( \alpha \) at \( \alpha(u_0) \) and \( \alpha(u_0) \) is a vertex of \( \alpha \).

Solution:

Denote by \( c = (a,b) \) the center of the circle \( C(x, y) = 0 \). Then the function \( g(u) = C(x(u), y(u)) = (x(u) - a)^2 + (y(u) - b)^2 - \lambda \) can be written as

\[
g(u) = (\alpha(u) - c) \cdot (\alpha(u) - c) - \lambda
\]

(a) Differentiating \( g(u) \), we obtain

\[
g'(u) = 2(\alpha(u) - c) \cdot \alpha'(u)
\]

which vanishes if and only if \( \alpha'(u) \) is orthogonal to \( \alpha(u) - c \). Note that \( \alpha(u) - c \) is a radius of the circle, and the vector \( \alpha'(u) \) is orthogonal to a radius if and only if it is tangent to the circle.

(b) Differentiating \( g'(u) \), we obtain

\[
g''(u) = 2(\alpha(u) - c) \cdot \alpha''(u) + 2||\alpha'(u)||^2
\]

Since \( \alpha(u) - c \) is orthogonal to \( \alpha'(u) \), it is collinear with \( \alpha''(u) \), namely, it is equal to \( \pm ||\alpha(u) - c||n \). Assume \( \kappa(u) > 0 \) (if \( \kappa(u) < 0 \) the computations are similar), then \( \alpha''(u) = -||\alpha(u) - c||n \). Thus, \( g''(u) = 0 \) if and only if

\[
-2||\alpha(u) - c||n \cdot \alpha''(u) + 2||\alpha'(u)||^2 = 0,
\]

which is equivalent to

\[
||\alpha(u) - c|| = \frac{||\alpha'(u)||^2}{n \cdot \alpha''(u)}
\]

The latter is equal to \( 1/\kappa(u) \) (see Exercise 2.2).

(c) Again, assume \( \kappa(u) > 0 \). According to (b), we can write

\[
g''(u) = -2||\alpha(u) - c||n \cdot \alpha''(u) + 2||\alpha'(u)||^2 =
\]

\[= -2||\alpha(u) - c||\kappa(u)||\alpha'(u)||^2 + 2||\alpha'(u)||^2 = 2||\alpha'(u)||^2(1 - \kappa(u)||\alpha(u) - c||)
\]

Differentiating this expression, we get

\[
g'''(u) = 4\alpha''(u) \cdot \alpha'(u)(1 - \kappa(u)||\alpha(u) - c||) + 2||\alpha'(u)||^2(-||\alpha(u) - c||\kappa(u) - ||\alpha(u) - c||\kappa'(u))
\]

Since the center \( c \) of the circle coincides with the center of curvature of \( \alpha \), the first summand is equal to zero. The derivative of \( ||\alpha(u) - c|| \) is also zero since \( \alpha'(u) \) is orthogonal to \( \alpha(u) - c \) (cf. (a) or Exercise 1.4(b)). Thus, \( g'''(u) = 0 \) if and only if \( \kappa'(u) = 0 \), or, equivalently, \( \alpha(u) \) is a vertex of \( \alpha \).