Durham University
Pavel Tumarkin

Michaelmas 2016

Differential Geometry III, Solutions 4 (Week 4)

Space curves - 1

4.1. Check that for two curves $\boldsymbol{\alpha}, \boldsymbol{\beta} : I \to \mathbb{R}^3$ holds

$$(\boldsymbol{\alpha}(u) \times \boldsymbol{\beta}(u))' = \boldsymbol{\alpha}'(u) \times \boldsymbol{\beta}(u) + \boldsymbol{\alpha}(u) \times \boldsymbol{\beta}'(u),$$

where $\boldsymbol{\alpha} \times \boldsymbol{\beta}$ is the cross-product in $\mathbb{R}^3$.

Solution: One can do a direct calculation in coordinates similar to Exercise 1.3. Alternatively, one can observe that coordinates of a cross-product are expressed via certain determinants which are miltilinear functions.

4.2. ($\star$) Find the curvature and torsion of the curve

$$\boldsymbol{\alpha}(u) = (au, bu^2, cu^3).$$

Solution:

We use Theorem 4.6. Since

$$\begin{aligned}
\boldsymbol{\alpha}'(u) &= (a, 2bu, 3cu^2),\\
\boldsymbol{\alpha}''(u) &= (0, 2b, 6cu),\\
\boldsymbol{\alpha}'''(u) &= (0, 0, 6c),
\end{aligned}$$

we have

$$\begin{aligned}
\kappa(u) &= \frac{\|\boldsymbol{\alpha}' \times \boldsymbol{\alpha}''\|}{\|\boldsymbol{\alpha}'\|^3} = \frac{\|(6bcu^2, -6acu, 2ab)\|}{(a^2 + 4b^2u^2 + 9c^2u^4)^{3/2}} = \frac{2(9b^2c^2u^4 + 9a^2c^2u^2 + a^2b^2)^{1/2}}{(a^2 + 4b^2u^2 + 9c^2u^4)^{3/2}},\\
\tau(u) &= \frac{-(6bcu^2, -6acu, 2ab) \cdot (0, 0, 6c)}{4(9b^2c^2u^4 + 9a^2c^2u^2 + a^2b^2)} = \frac{-3abc}{(9b^2c^2u^4 + 9a^2c^2u^2 + a^2b^2)}
\end{aligned}$$

4.3. ($\star$) Assume that $\boldsymbol{\alpha} : I \to \mathbb{R}^3$ is a regular space curve parametrized by arc length.

(a) Determine all regular curves with vanishing curvature κ.

Hint: use Theorem 4.6

(b) Show that if the torsion τ of $\boldsymbol{\alpha}$ vanishes, then the trace of $\boldsymbol{\alpha}$ lies in a plane.

Hint: do NOT use Theorem 4.6

Solution:

(a) By Theorem 4.6, $\kappa(s) = 0$ if and only if $\boldsymbol{\alpha}'(s) \times \boldsymbol{\alpha}''(s) = 0$. Note that since $\boldsymbol{\alpha}$ is regular, $\boldsymbol{\alpha}'(s) \neq 0$. If $\boldsymbol{\alpha}''(s) \equiv 0$, then $\boldsymbol{\alpha}'(s) = (a, b, c)$ for some constants $a, b, c \in \mathbb{R}$, and thus

$$\boldsymbol{\alpha}(s) = \boldsymbol{\alpha}_0 + s(a, b, c)$$

is a line.

Assume now that $\boldsymbol{\alpha}''(s) \neq 0$ at some point s (and thus, in some neighborhood of s). Then the unit normal $\boldsymbol{n}(s)$ is the unit vector defined by

$$\boldsymbol{n}(s) = \frac{\boldsymbol{\alpha}''(s)}{\|\boldsymbol{\alpha}''(s)\|},$$

so,

$$\kappa(s) = \frac{\|\boldsymbol{\alpha}''(s)\|}{\|\boldsymbol{n}(s)\|} = \|\boldsymbol{\alpha}''(s)\| \neq 0,$$

which leads to a contradiction.

Therefore, the only regular curve with zero curvature is a line.

(b) By Serret-Frenet equations, $\boldsymbol{b}' = \tau \boldsymbol{n}$. Thus, if $\tau \equiv 0$, then $\boldsymbol{b}$ is constant. In particular, all the planes spanned by $\boldsymbol{t}(s)$ and $\boldsymbol{n}(s)$ are parallel. We want to show that they all coincide.

Choose any s_0, and consider the function

$$f(s) = \boldsymbol{b} \cdot (\boldsymbol{\alpha}(s) - \boldsymbol{\alpha}(s_0))$$

The derivative of this function is

$$f'(s) = \boldsymbol{b}' \cdot (\boldsymbol{\alpha}(s) - \boldsymbol{\alpha}(s_0)) + \boldsymbol{b} \cdot (\boldsymbol{\alpha}(s) - \boldsymbol{\alpha}(s_0))' = \boldsymbol{0} \cdot (\boldsymbol{\alpha}(s) - \boldsymbol{\alpha}(s_0)) + \boldsymbol{b} \cdot \boldsymbol{\alpha}'(s) = 0$$

which implies that $f(s)$ is constant. Since for $f(s_0) = 0$, we see that $\boldsymbol{\alpha}(s)$ satisfies

$$\boldsymbol{b} \cdot (\boldsymbol{\alpha}(s) - \boldsymbol{\alpha}_0) = 0$$

for constant vectors $\boldsymbol{b}$ and $\boldsymbol{\alpha}_0$. The equation above is an equation of a plane in $\mathbb{R}^3$.

4.4. Assume that $\boldsymbol{\alpha}(s) = (x(s), y(s), 0)$, i.e., the trace of $\boldsymbol{\alpha}$ lies in the plane $z = 0$. Calculate the curvature κ of $\boldsymbol{\alpha}$ and its torsion τ. What is the relation of the curvature κ of the space curve $\boldsymbol{\alpha}$ and the (signed) curvature $\overline{\kappa}$ of the plane curve $\overline{\boldsymbol{\alpha}} : I \to \mathbb{R}^2$ defined by $\overline{\boldsymbol{\alpha}}(s) = (x(s), y(s))$ (i.e., the projection of the space curve $\boldsymbol{\alpha}$ to the plane $z = 0$)?

Solution:

Since $\boldsymbol{\alpha}$ lies in the plane $z = 0$, the tangent and normal vectors also lie in the plane, so the binormal vector is constant. Using the equation $\boldsymbol{b}' = \tau \boldsymbol{n}$ we see that $\tau \equiv 0$. The curvature of $\boldsymbol{\alpha}$ is clearly the absolute value of the curvature of $\overline{\boldsymbol{\alpha}}$.

4.5. Consider the regular curve given by

$$\boldsymbol{\alpha}(s) = \left(a \cos \frac{s}{c}, a \sin \frac{s}{c}, b\frac{s}{c}\right), \qquad s \in \mathbb{R},$$

where $a, b, c > 0$ and $c^2 = a^2 + b^2$. The curve $\boldsymbol{\alpha}$ is called a *helix*.

(a) Show that the trace of $\boldsymbol{\alpha}$ lies on the cylinder $x^2 + y^2 = a^2$.

(b) Show that $\boldsymbol{\alpha}$ is parametrized by arc length.

(c) Determine the curvature and torsion of $\boldsymbol{\alpha}$ (and notice that they are both constant).

(d) Determine the equation of the plane through $\boldsymbol{n}(s)$ and $\boldsymbol{t}(s)$ at each point of $\boldsymbol{\alpha}$ (this plane is called the *osculating plane*).

(e) Show that the line through $\boldsymbol{\alpha}(s)$ in direction $\boldsymbol{n}(s)$ meets the axis of the cylinder orthogonally.

(f) Show that the tangent lines to $\boldsymbol{\alpha}$ make a constant angle with the axis of the cylinder.

Solution:

(a)

$$x(s)^2 + y(s)^2 = a^2 \left(\cos^2 \frac{s}{c} + \sin^2 \frac{s}{c}\right) = a^2,$$

i.e., the trace of $\boldsymbol{\alpha}$ lies on the cylinder $x^2 + y^2 = a^2$.

(b) We have

$$\boldsymbol{\alpha}'(s) = \left(-\frac{a}{c}\sin\frac{s}{c}, \frac{a}{c}\cos\frac{s}{c}, \frac{b}{c}\right),$$

which implies

$$\|\boldsymbol{\alpha}'(s)\|^2 = \frac{a^2}{c^2} + \frac{b^2}{c^2} = 1.$$

This shows that $\boldsymbol{\alpha}$ is unit speed.

(c) We have

$$\begin{aligned}
\boldsymbol{\alpha}''(s) &= \left(-\frac{a}{c^2}\cos\frac{s}{c}, -\frac{a}{c^2}\sin\frac{s}{c}, 0\right), \\
\boldsymbol{\alpha}'''(s) &= \left(\frac{a}{c^3}\sin\frac{s}{c}, -\frac{a}{c^3}\cos\frac{s}{c}, 0\right).
\end{aligned}$$

This implies

$$\boldsymbol{\alpha}'(s) \times \boldsymbol{\alpha}''(s) = \left(\frac{ab}{c^3}\sin\frac{s}{c}, -\frac{ab}{c^3}\cos\frac{s}{c}, \frac{a^2}{c^3}\right).$$

We conclude that

$$\|\boldsymbol{\alpha}'(s) \times \boldsymbol{\alpha}''(s)\|^2 = \frac{a^2(a^2+b^2)}{c^6} = \frac{a^2}{c^4},$$

i.e.,

$$\kappa = \frac{\|\boldsymbol{\alpha}' \times \boldsymbol{\alpha}''\|}{\|\boldsymbol{\alpha}'\|^3} = \frac{a}{c^2}.$$

Moreover, we have

$$(\boldsymbol{\alpha}'(s) \times \boldsymbol{\alpha}''(s)) \cdot \boldsymbol{\alpha}'''(s) = \frac{a^2 b}{c^6}\sin^2\frac{s}{c} + \frac{a^2 b}{c^6}\cos^2\frac{s}{c} = \frac{a^2 b}{c^6}.$$

This implies that

$$\tau = -\frac{a^2 b}{c^6} \cdot \frac{c^4}{a^2} = -\frac{b}{c^2}.$$

(d) The osculating plane is orthogonal to the binormal vector $\boldsymbol{b}(s)$, and thus to $\boldsymbol{\alpha}'(s) \times \boldsymbol{\alpha}''(s)$ which is collinear to $\boldsymbol{b}(s)$. We have already computed in (c) that

$$\boldsymbol{\alpha}'(s) \times \boldsymbol{\alpha}''(s) = \left(\frac{ab}{c^3}\sin\frac{s}{c}, -\frac{ab}{c^3}\cos\frac{s}{c}, \frac{a^2}{c^3}\right).$$

Therefore, the equation of the osculating plane at $\boldsymbol{\alpha}(s) = (x(s), y(s), z(s))$ can be written as

$$\frac{ab}{c^3}\sin\frac{s}{c}(x - x(s)) - \frac{ab}{c^3}\cos\frac{s}{c}(y - y(s)) + \frac{a^2}{c^3}(z - z(s)) = 0.$$

After plugging in the explicit expressions for $\boldsymbol{\alpha}(s)$ and multiplying by c^3/a we obtain

$$xb\sin\frac{s}{c} - yb\cos\frac{s}{c} + az - ab\frac{s}{c} = 0$$

(e) Normalizing the expression for $\boldsymbol{\alpha}''(s)$ obtained in (c), we see that $\boldsymbol{n}(s) = \left(-\cos\frac{s}{c}, -\sin\frac{s}{c}, 0\right)$. Since the z-coordinate of $\boldsymbol{n}(s)$ is zero, $\boldsymbol{n}(s)$ is orthogonal to the z-axis (which is also the axis of the cylinder). Note also that $a\boldsymbol{n}(s)$ is a projection of $-\boldsymbol{\alpha}(s)$ onto the horizontal plane, so the line $\boldsymbol{\alpha}(s) + u\boldsymbol{n}(s)$ meets the z-axis at $u = a$.

(f) To find the cosine of the angle, we need to compute the dot product of the unit tangent vector and the unit vector in the direction of the axis of the cylinder. The latter has coordinates $(0, 0, 1)$, so the cosine is equal to

$$(0,0,1) \cdot \left(-\frac{a}{c}\sin\frac{s}{c}, \frac{a}{c}\cos\frac{s}{c}, \frac{b}{c}\right) = \frac{b}{c}$$

which is constant.