Durham University
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Differential Geometry III, Homework 5 (Week 5)

Due date for starred problems: **Thursday, November 24**.

Space curves - 2

5.1. ($\star$) A curve $\boldsymbol{\alpha} : I \to \mathbb{R}^3$ is called a *(generalized) helix* if its tangent lines make a constant angle with some fixed direction in $\mathbb{R}^3$.

(a) Prove that the curve

$$\boldsymbol{\alpha}(s) = \Big(\frac{a}{c}\int_{s_0}^{s} \sin\vartheta(v)\,\mathrm{d}v, \frac{a}{c}\int_{s_0}^{s} \cos\vartheta(v)\,\mathrm{d}v, \frac{b}{c}s\Big),$$

with $s_0 \in I$, $c^2 = a^2 + b^2$, $a \neq 0$, $b \neq 0$ and $\vartheta'(s) > 0$ is a (generalized) helix.

(b) Assume that $\boldsymbol{\alpha} : I \to \mathbb{R}^3$ is a regular curve with $\tau(s) \neq 0$ for all $s \in I$. Prove that $\boldsymbol{\alpha}$ is a (generalized) helix if and only if κ/τ is constant.

5.2. Let $\boldsymbol{\alpha}$, $\boldsymbol{\beta}$ be regular curves in $\mathbb{R}^3$ such that, for each u, the principal normals $\boldsymbol{n}_{\boldsymbol{\alpha}}(u)$ and $\boldsymbol{n}_{\boldsymbol{\beta}}(u)$ are parallel. Prove that the angle between $\boldsymbol{t}_{\boldsymbol{\alpha}}(u)$ and $\boldsymbol{t}_{\boldsymbol{\beta}}(u)$ is independent of u. Prove also that if the line through $\boldsymbol{\alpha}(u)$ in direction $\boldsymbol{n}_{\boldsymbol{\alpha}(u)}$ coincides with the line through $\boldsymbol{\beta}(u)$ in direction $\boldsymbol{n}_{\boldsymbol{\beta}(u)}$ then

$$\boldsymbol{\beta}(u) = \boldsymbol{\alpha}(u) + r\boldsymbol{n}_{\boldsymbol{\alpha}}(u)$$

for some real number r.

5.3. ($\star$) Let $\boldsymbol{\alpha}$ be the curve in $\mathbb{R}^3$ given by

$$\boldsymbol{\alpha}(u) = e^u(\cos u, \sin u, 1), \qquad u \in \mathbb{R}.$$

If $0 < \lambda_0 < \lambda_1$, find the length of the segment of $\boldsymbol{\alpha}$ which lies between the planes $z = \lambda_0$ and $z = \lambda_1$. Show also that the curvature and torsion of $\boldsymbol{\alpha}$ are both inversely proportional to e^u.

5.4. Let $\boldsymbol{\alpha}$ be a curve parametrized by arc length with nowhere vanishing curvature κ and torsion τ. Show that if the trace of $\boldsymbol{\alpha}$ lies on a sphere then

$$\frac{\tau}{\kappa} = \Big(\frac{\kappa'}{\tau\kappa^2}\Big)'.$$

Is the converse true?

5.5. Let $\boldsymbol{\alpha}$ be a regular curve parametrized by arc length with $\kappa > 0$ and $\tau \neq 0$. Denote by $\boldsymbol{n}$ and $\boldsymbol{b}$ the principal normal and the binormal of $\boldsymbol{\alpha}$.

(a) If $\boldsymbol{\alpha}$ lies on a sphere with center $\boldsymbol{c} \in \mathbb{R}^3$ and radius $r > 0$, show that

$$\boldsymbol{\alpha} - \boldsymbol{c} = -\rho\boldsymbol{n} - \rho'\sigma\boldsymbol{b},$$

where $\rho = 1/\kappa$ and $\sigma = -1/\tau$. Deduce that $r^2 = \rho^2 + (\rho'\sigma)^2$.

(b) Conversely, if $\rho^2 + (\rho'\sigma)^2$ has constant value r^2 and $\rho' \neq 0$, show that $\boldsymbol{\alpha}$ lies on a sphere of radius r.

Hint: Show that the curve $\boldsymbol{\alpha} + \rho\boldsymbol{n} + \rho'\sigma\boldsymbol{b}$ is constant.