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Differential Geometry III, Solutions 8 (Week 8)

Tangent plane

8.1. (a) Let $\boldsymbol{x} : U \to S$ be a local parametrization of a surface S in some neighborhood of a point $\boldsymbol{p} = (x_0, y_0, z_0) \in S$. Show that the tangent plane to S at $\boldsymbol{p}$ has an equation

$$\left(\frac{\partial \boldsymbol{x}}{\partial u}(\boldsymbol{p}) \times \frac{\partial \boldsymbol{x}}{\partial v}(\boldsymbol{p})\right) \cdot (x - x_0, y - y_0, z - z_0) = 0$$

(b) Let $f : \mathbb{R}^3 \to \mathbb{R}$ be a smooth function, and let $c \in f(\mathbb{R}^3)$ be a regular value of f. Show that the tangent plane of a regular surface

$$S = \{(x, y, z) \,|\, f(x, y, z) = c\}$$

at the point $\boldsymbol{p} = (x_0, y_0, z_0) \in S$ has equation

$$\frac{\partial f}{\partial x}(\boldsymbol{p})(x - x_0) + \frac{\partial f}{\partial y}(\boldsymbol{p})(y - y_0) + \frac{\partial f}{\partial z}(\boldsymbol{p})(z - z_0) = 0$$

Solution:

(a) By definition, the tangent plane to S at $\boldsymbol{p} \in S$ is spanned by vectors $\frac{\partial \boldsymbol{x}}{\partial u}(\boldsymbol{p})$ and $\frac{\partial \boldsymbol{x}}{\partial v}(\boldsymbol{p})$ and passes through $\boldsymbol{p}$. Let Π be the plane defined by the equation above. Since $\frac{\partial \boldsymbol{x}}{\partial u}(\boldsymbol{p}) \times \frac{\partial \boldsymbol{x}}{\partial v}(\boldsymbol{p})$ is orthogonal to both $\frac{\partial \boldsymbol{x}}{\partial u}(\boldsymbol{p})$ and $\frac{\partial \boldsymbol{x}}{\partial v}(\boldsymbol{p})$, the both partial derivatives lie in Π. Now, the point $\boldsymbol{p} = (x_0, y_0, z_0)$ itself clearly satisfies the equation.

(b) If $\boldsymbol{\alpha} : (-\varepsilon, \varepsilon) \to S$ is any curve with $\boldsymbol{\alpha}(0) = \boldsymbol{p}$, then $f(\boldsymbol{\alpha}(u)) \equiv c$. Differentiating, we obtain

$$\nabla f(p) \cdot \boldsymbol{\alpha}'(0) = 0,$$

which implies that the tangent plane is orthogonal to the gradient $\nabla f(p) = \left(\frac{\partial f}{\partial x}(\boldsymbol{p}), \frac{\partial f}{\partial y}(\boldsymbol{p}), \frac{\partial f}{\partial z}(\boldsymbol{p})\right)$.

8.2. ($\star$) Show that the tangent plane of one-sheeted hyperboloid $x^2 + y^2 - z^2 = 1$ at point $(x, y, 0)$ is parallel to the z-axis.

Solution:

Using Exercise 8.1(b), we see that the tangent plane at point $(x_0, y_0, 0)$ of the hyperboloid has an equation

$$x_0(x - x_0) + y_0(y - y_0) = 0$$

which is clearly parallel to z-axis.

8.3. Let $f : \mathbb{R} \to \mathbb{R}$ be a smooth function. Define a surface S as

$$S = \{(x, y, z) \,|\, xf(y/x) - z = 0,\ x \neq 0\}$$

Show that all tangent planes of S pass through the origin $(0, 0, 0)$.

Solution:

The surface is the graph of a smooth function $z = xf(y/x)$, so it has a parametrization

$$\boldsymbol{x}(x,y) = (x, y, xf(y/x))$$

First, we compute $\frac{\partial \boldsymbol{x}}{\partial x}$ and $\frac{\partial \boldsymbol{x}}{\partial y}$, and then use Exercise 8.1(a).

$$\begin{aligned}\frac{\partial \boldsymbol{x}}{\partial x}(x,y) &= \left(1, 0, f\left(\frac{y}{x}\right) - \frac{y}{x} f'\left(\frac{y}{x}\right)\right), \\ \frac{\partial \boldsymbol{x}}{\partial y}(x,y) &= \left(0, 1, f'\left(\frac{y}{x}\right)\right)\end{aligned}$$

Thus,

$$\frac{\partial \boldsymbol{x}}{\partial x} \times \frac{\partial \boldsymbol{x}}{\partial y}(x,y) = \left(-f\left(\frac{y}{x}\right) + \frac{y}{x} f'\left(\frac{y}{x}\right), -f'\left(\frac{y}{x}\right), 1\right),$$

and an equation of the tangent plane at $(x_0, y_0, z_0) \in S$ is

$$\left(-f\left(\frac{y_0}{x_0}\right) + \frac{y_0}{x_0} f'\left(\frac{y_0}{x_0}\right), -f'\left(\frac{y_0}{x_0}\right), 1\right) \cdot (x - x_0, y - y_0, z - z_0) = 0$$

This plane passes through the origin if and only if

$$\left(-f\left(\frac{y_0}{x_0}\right) + \frac{y_0}{x_0} f'\left(\frac{y_0}{x_0}\right), -f'\left(\frac{y_0}{x_0}\right), 1\right) \cdot (x_0, y_0, z_0) = 0$$

Indeed, taking into account that

$$f\left(\frac{y_0}{x_0}\right) = \frac{z_0}{x_0},$$

we have

$$\left(-f\left(\frac{y_0}{x_0}\right) + \frac{y_0}{x_0} f'\left(\frac{y_0}{x_0}\right), -f'\left(\frac{y_0}{x_0}\right), 1\right) \cdot (x_0, y_0, z_0) = -\frac{z_0}{x_0} x_0 + y_0 f'\left(\frac{y_0}{x_0}\right) - y_0 f'\left(\frac{y_0}{x_0}\right) + z_0 = 0$$

8.4. Let $U \subset \mathbb{R}^2$ be open, and let S_1 and S_2 be two regular surfaces with parametrizations $\boldsymbol{x} : U \to S_1$ and $\boldsymbol{y} : U \to S_2$. Define a map $\boldsymbol{\varphi} = \boldsymbol{y} \circ \boldsymbol{x}^{-1} : S_1 \to S_2$. Let $\boldsymbol{p} \in S_1$, $\boldsymbol{w} \in T_{\boldsymbol{p}} S_1$, and let $\boldsymbol{\alpha} : (-\varepsilon, \varepsilon) \to S_1$ be an arbitrary regular curve in S_1 such that $\boldsymbol{p} = \boldsymbol{\alpha}(0)$ and $\boldsymbol{\alpha}'(0) = \boldsymbol{w}$. Define $\boldsymbol{\beta} : (-\varepsilon, \varepsilon) \to S_2$ as $\boldsymbol{\beta} = \boldsymbol{\varphi} \circ \boldsymbol{\alpha}$.

(a) Show that $\boldsymbol{\beta}'(0)$ does not depend on the choice of $\boldsymbol{\alpha}$.

(b) Show that the map $\mathrm{d}_{\boldsymbol{p}}\boldsymbol{\varphi} : T_{\boldsymbol{p}} S_1 \to T_{\boldsymbol{\varphi}(\boldsymbol{p})} S_2$ defined by $\mathrm{d}_{\boldsymbol{p}}\boldsymbol{\varphi}(\boldsymbol{w}) = \boldsymbol{\beta}'(0)$ is linear.

Solution:

(a) Define a curve $\boldsymbol{\gamma} : (-\varepsilon, \varepsilon) \to U$ by $\boldsymbol{\alpha} = \boldsymbol{x} \circ \boldsymbol{\gamma}$, and define $\boldsymbol{q} \in U$ by $\boldsymbol{x}(\boldsymbol{q}) = \boldsymbol{p}$. Then, by the chain rule,

$$\boldsymbol{w} = \boldsymbol{\alpha}'(0) = (\boldsymbol{x} \circ \boldsymbol{\gamma})'(0) = \mathrm{d}_{\gamma(0)}\boldsymbol{x}(\gamma'(0)) = \mathrm{d}_{\boldsymbol{q}}\boldsymbol{x}(\gamma'(0))$$

Thus,

$$\boldsymbol{\gamma}'(0) = (\mathrm{d}_{\boldsymbol{q}}\boldsymbol{x})^{-1}(\boldsymbol{w}),$$

where by $(\mathrm{d}_{\boldsymbol{q}}\boldsymbol{x})^{-1}$ we mean the *left* inverse of $\mathrm{d}_{\boldsymbol{q}}\boldsymbol{x}$, namely, a linear map from $\mathbb{R}^3$ to $\mathbb{R}^2$ satisfying $(\mathrm{d}_{\boldsymbol{q}}\boldsymbol{x})^{-1} \circ \mathrm{d}_{\boldsymbol{q}}\boldsymbol{x} = \mathrm{id}_{\mathbb{R}^2}$ (notice that $\mathrm{d}_{\boldsymbol{q}}\boldsymbol{x}$ has no inverse since it is a linear map from $\mathbb{R}^2$ to $\mathbb{R}^3$). In particular, we see that $\boldsymbol{\gamma}'(0)$ does not depend on the choice of $\boldsymbol{\alpha}$ but on the vector $\boldsymbol{w}$ only.

Now, we can write

$$\boldsymbol{\beta} = \boldsymbol{y} \circ \boldsymbol{\gamma},$$

and differentiating this we get

$$\boldsymbol{\beta}'(0) = (\boldsymbol{y} \circ \boldsymbol{\gamma})'(0) = \mathrm{d}_{\gamma(0)}\boldsymbol{y}(\boldsymbol{\gamma}'(0)) = \mathrm{d}_{\boldsymbol{q}}\boldsymbol{y}(\boldsymbol{\gamma}'(0))$$

Therefore, $\boldsymbol{\beta}'(0)$ is completely defined by $\mathrm{d}_{\boldsymbol{q}}\boldsymbol{y}$ and $\boldsymbol{\gamma}'(0)$ which do not depend on the choice of $\boldsymbol{\alpha}$.

(b) As we have seen in (a),

$$\mathrm{d}_{\boldsymbol{p}}\boldsymbol{\varphi}(\boldsymbol{w}) = \boldsymbol{\beta}'(0) = \mathrm{d}_{\boldsymbol{q}}\boldsymbol{y}(\gamma'(0)) = \mathrm{d}_{\boldsymbol{q}}\boldsymbol{y}((\mathrm{d}_{\boldsymbol{q}}\boldsymbol{x})^{-1}(w)) = (\mathrm{d}_{\boldsymbol{q}}\boldsymbol{y} \circ (\mathrm{d}_{\boldsymbol{q}}\boldsymbol{x})^{-1})(\boldsymbol{w}),$$

which implies

$$\mathrm{d}_{\boldsymbol{p}}\boldsymbol{\varphi} = \mathrm{d}_{\boldsymbol{q}}\boldsymbol{y} \circ (\mathrm{d}_{\boldsymbol{q}}\boldsymbol{x})^{-1}$$

which is clearly linear as a composition of two linear maps.

8.5. Let $\boldsymbol{\alpha} : I \to \mathbb{R}^3$ be a regular curve with nonzero curvature parametrized by arc length. Recall that a *canal surface* (or *tubular surface*) S is a surface parametrized by

$$\boldsymbol{x}(u,v) = \boldsymbol{\alpha}(u) + r(\boldsymbol{n}(u)\cos v + \boldsymbol{b}(u)\sin v),$$

where $\boldsymbol{n}$ and $\boldsymbol{b}$ are unit normal and binormal vectors, and $r > 0$ is a sufficiently small constant. Find the equation of the tangent plane to S at $\boldsymbol{x}(u,v)$. In particular, show that the tangent plane at $\boldsymbol{x}(u,v)$ is parallel to $\boldsymbol{\alpha}'(u)$.

Solution: We use Exercise 8.1(a) to compute an equation of the tangent plane.

$$\frac{\partial \boldsymbol{x}}{\partial u}(u,v) = \boldsymbol{\alpha}'(u) + r(\boldsymbol{n}'(u)\cos v + \boldsymbol{b}'(u)\sin v) = \boldsymbol{t} + r(-\kappa\boldsymbol{t} - \tau\boldsymbol{b})\cos v + r\tau\boldsymbol{n}\sin v =$$
$$= \boldsymbol{t}(1 - r\kappa\cos v) + \boldsymbol{n}(r\tau\sin v) + \boldsymbol{b}(-r\tau\cos v),$$

and

$$\frac{\partial \boldsymbol{x}}{\partial v}(u,v) = r(\boldsymbol{n}(u)(-\sin v) + \boldsymbol{b}(u)\cos v) = \boldsymbol{n}(-r\sin v) + \boldsymbol{b}(r\cos v)$$

Now, computing the cross-product, we get

$$\left(\frac{\partial \boldsymbol{x}}{\partial u} \times \frac{\partial \boldsymbol{x}}{\partial v}\right)(u,v) = -r(1 - r\kappa\cos v)(\boldsymbol{n}(u)\cos v + \boldsymbol{b}(u)\sin v)$$

An equation of the tangent plane to S at point $\boldsymbol{x}(u_0, v_0)$ with respect to variable $\boldsymbol{q} \in \mathbb{R}^3$ can be written as

$$(\boldsymbol{n}(u_0)\cos v_0 + \boldsymbol{b}(u_0)\sin v_0) \cdot (\boldsymbol{q} - (\boldsymbol{\alpha}(u_0) + r(\boldsymbol{n}(u_0)\cos v_0 + \boldsymbol{b}(u_0)\sin v_0)) = 0$$

Since $\boldsymbol{n}(u_0)\cos v_0 + \boldsymbol{b}(u_0)\sin v_0$ is a unit vector, this is equivalent to

$$(\boldsymbol{n}(u_0)\cos v_0 + \boldsymbol{b}(u_0)\sin v_0) \cdot (\boldsymbol{q} - \boldsymbol{\alpha}(u_0)) = r$$

In particular, the vector $\boldsymbol{n}(u_0)\cos v_0 + \boldsymbol{b}(u_0)\sin v_0$ is orthogonal to $\boldsymbol{t}(u_0)$ as a linear combination of $\boldsymbol{n}(u_0)$ and $\boldsymbol{b}(u_0)$, so the plane is parallel to $\boldsymbol{t}(u_0)$.