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Differential Geometry III

2 Regular curves in $\mathbb{R}^n$

Definition 2.1.

(a) A *smooth curve* in $\mathbb{R}^n$ is a smooth (that is, infinitely differentiable) map

$$\boldsymbol{\alpha}\colon I \to \mathbb{R}^n,$$

where I is an open interval of $\mathbb{R}$ (so I could be (a,b) or $(-\infty,b)$ or $(a,+\infty)$ or $\mathbb{R}$).

(b) The *image*, $\boldsymbol{\alpha}(I)$, of I under $\boldsymbol{\alpha}$ is called the *trace* of $\boldsymbol{\alpha}$. The variable $u \in I$ is called the *parameter* of $\boldsymbol{\alpha}$.

(c) If we write

$$\boldsymbol{\alpha}(u) = (\alpha_1(u), \alpha_2(u), \ldots, \alpha_n(u))$$

then each $\alpha_i\colon I \to \mathbb{R}$ is smooth. The vector

$$\boldsymbol{\alpha}'(u) = (\alpha_1'(u), \alpha_2'(u), \ldots, \alpha_n'(u))$$

is the *tangent vector* to $\boldsymbol{\alpha}$ at $\boldsymbol{\alpha}(u)$.

(d) The curve $\boldsymbol{\alpha}$ is *regular* if $\boldsymbol{\alpha}'(u) \neq \mathbf{0} = (0,\ldots,0)$ for all $u \in I$. The curve $\boldsymbol{\alpha}$ is *singular* at $\boldsymbol{\alpha}(u)$ if $\boldsymbol{\alpha}'(u) = \mathbf{0}$.

(e) If $\boldsymbol{\alpha}$ is a regular curve, we define the unit tangent vector

$$\boldsymbol{t}(u) = \frac{\boldsymbol{\alpha}'(u)}{\|\boldsymbol{\alpha}'(u)\|}.$$

If we want to stress that $\boldsymbol{t}$ is the unit tangent vector of the curve $\boldsymbol{\alpha}$, we also write $\boldsymbol{t}_{\boldsymbol{\alpha}}$.

(f) If $\|\boldsymbol{\alpha}'(u)\| = 1$ for all $u \in I$ then $\boldsymbol{\alpha}$ is called *unit speed.*

Example 2.2.

(a) *The unit circle.* $\boldsymbol{\alpha}\colon \mathbb{R} \longrightarrow \mathbb{R}^2$, $\boldsymbol{\alpha}(u) = (\cos u, \sin u)$. $\boldsymbol{\alpha}$ is smooth and unit speed.

(b) *The helix.* $\boldsymbol{\alpha} : \mathbb{R} \to \mathbb{R}^3$, $\boldsymbol{\alpha}(u) = (\cos u, \sin u, u)$. $\boldsymbol{\alpha}$ is smooth and regular.

(c) *The cusp.* $\boldsymbol{\alpha}\colon \mathbb{R} \longrightarrow \mathbb{R}^2$, $\boldsymbol{\alpha} = (u^3, u^2)$ so $\boldsymbol{\alpha}$ is smooth. But $\boldsymbol{\alpha}'(s) = (3u^2, 2u)$, so $\boldsymbol{\alpha}'(0) = (0,0)$.

(d) *The node.* $\boldsymbol{\alpha} : \mathbb{R} \to \mathbb{R}^2$, $\boldsymbol{\alpha}(u) = (u^3 - u, u^2 - 1)$. $\boldsymbol{\alpha}$ is smooth and regular but not injective, since $\boldsymbol{\alpha}(-1) = \boldsymbol{\alpha}(1)$.

Definition 2.3. Let $\boldsymbol{\alpha}\colon I \longrightarrow \mathbb{R}^n$ be a smooth and regular curve. A *change of parameter* for $\boldsymbol{\alpha}$ is a function $h\colon J \longrightarrow I$ where J is an open interval of $\mathbb{R}$ satisfying

(a) h is smooth;

(b) $h'(t) \neq 0$ for all $t \in J$;

(c) $h(J) = I$.

Remark. $\tilde{\boldsymbol{\alpha}} = \boldsymbol{\alpha} \circ h\colon J \longrightarrow \mathbb{R}^n$ is a smooth curve with the same trace as $\boldsymbol{\alpha}$.

Example 2.4. In the Example 2.2(a) take $J = \mathbb{R}$, $h(v) = 2v$. Then

$$\tilde{\boldsymbol{\alpha}}(v) = (\boldsymbol{\alpha} \circ h)(v) = \boldsymbol{\alpha}(2v) = (\cos 2v, \sin 2v).$$

Definition 2.5. The *arc length* of a curve $\boldsymbol{\alpha}\colon I \longrightarrow \mathbb{R}^n$, measured from a point $\boldsymbol{\alpha}(u_0)$ for some $u_0 \in I$, is

$$\ell(u) := \int_{u_0}^{u} \|\boldsymbol{\alpha}'(v)\| \,\mathrm{d}v.$$

Remark. If $\boldsymbol{\alpha}$ is unit speed ($\|\boldsymbol{\alpha}'(u)\| = 1$), then

$$\ell(u) = \int_{u_0}^{u} \|\boldsymbol{\alpha}'(s)\| \,\mathrm{d}s = u - u_0.$$

So the parameter u measures the arc length (up to an additive constant) and is called *arc length parameter*, $\boldsymbol{\alpha}$ is *parametrized by arc length.*

Proposition 2.6. Let $\boldsymbol{\alpha}\colon I \longrightarrow \mathbb{R}^n$ be a smooth and regular curve. Choose $u_o \in I$, and let $\ell\colon I \longrightarrow \mathbb{R}$ be the arc length of $\boldsymbol{\alpha}$ w.r. to u_0. Define $J = \ell(I)$. Then ℓ^{-1} is a parameter change, and

$$\boldsymbol{\beta} = \boldsymbol{\alpha} \circ \ell^{-1}\colon J \longrightarrow \mathbb{R}^n$$

is parametrized by arc length.

Example 2.7. *The catenary.*

$$\boldsymbol{\alpha}\colon \mathbb{R} \longrightarrow \mathbb{R}^2, \quad \boldsymbol{\alpha}(u) = (u, \cosh u) \quad \Rightarrow \boldsymbol{\alpha}'(u) = (1, \sinh u)$$

$\boldsymbol{\alpha}$ is regular, $\|\boldsymbol{\alpha}'(u)\| = \sqrt{1 + \sinh^2 u} = \cosh u$,

$$s = \ell(u) = \int_0^u \|\boldsymbol{\alpha}'(t)\| \,\mathrm{d}t = \int_0^u \cosh t \,\mathrm{d}t = \sinh u$$

where we fixed $u_0 = 0$, and thus $u = \ell^{-1}(s) = \sinh^{-1} s$. So the arc-length parametrization of the catenary is

$$\boldsymbol{\beta} = \boldsymbol{\alpha}(\ell^{-1}(s)) = \big(\ln(s + \sqrt{s^2 + 1}), \cosh(\ln(s + \sqrt{s^2 + 1}))\big).$$

3 Plane curves

3.1 Tangent and normal vectors. Curvature

Let $\boldsymbol{\alpha}\colon I \longrightarrow \mathbb{R}^2$ be a plane curve parametrized by arc length, i.e., $\boldsymbol{\alpha}'(s) = \mathbf{t}(s)$ is a unit vector.

Definition 3.1. The *unit normal vector* $\mathbf{n}(s)$ is the vector obtained by rotating $\mathbf{t}(s)$ anticlockwise through $\pi/2$.

In coordinates, if $\boldsymbol{\alpha}(s) = (x(s), y(s))$, then

$$\mathbf{t}(s) = (x'(s), y'(s)), \qquad \mathbf{n}(s) = (-y'(s), x'(s))$$

Remark. Differentiating the equation $1 = \|\mathbf{t}(s)\|^2 = \mathbf{t}(s) \cdot \mathbf{t}(s)$ gives

$$0 = \mathbf{t}'(s) \cdot \mathbf{t}(s) + \mathbf{t}(s) \cdot \mathbf{t}'(s) = 2\mathbf{t}'(s) \cdot \mathbf{t}(s).$$

In particular, $\mathbf{t}(s)$ and $\mathbf{t}'(s)$ are orthogonal, and hence $\mathbf{t}'(s)$ is parallel to the normal vector $\mathbf{n}(s)$ (which is also orthogonal to $\mathbf{t}(s)$). (Note that we use here the fact that we are in $\mathbb{R}^2$, otherwise the last conclusion that $\mathbf{t}'(s)$ is parallel to $\mathbf{n}(s)$ is not true!)

Definition 3.2. The *(signed) curvature* $\kappa(s)$ of a plane curve $\boldsymbol{\alpha}\colon I \longrightarrow \mathbb{R}^2$ is defined by $\mathbf{t}'(s) = \kappa(s)\mathbf{n}(s)$.

Remark. A way to compute: $\mathbf{n}(s) \cdot \mathbf{t}'(s) = \kappa(s)\mathbf{n}(s) \cdot \mathbf{n}(s) = \kappa(s)$ (since $\mathbf{n}(s)$ is a unit vector), so we have

$$\kappa(s) = \mathbf{n}(s) \cdot \mathbf{t}'(s)$$

If $\boldsymbol{\alpha}$ is given by $\boldsymbol{\alpha}(s) = (x(s), y(s))$, where s is the arc length, then

$$\kappa(s) = -y'(s)x''(s) + x'(s)y''(s),$$

provided the curve is parametrized by arc length.

Example 3.3. (a) *Lines.* $\kappa(s) \equiv 0$.

(b) *Circles.* $\kappa(s) \equiv 1/r$ for a circle of radius r.

Proposition 3.4. Let $\boldsymbol{\alpha}\colon I \longrightarrow \mathbb{R}^2$, $\boldsymbol{\alpha}(u) = (x(u), y(u))$, be a regular curve (not necessarily parametrized by arc length). Then

$$\kappa = \frac{x'y'' - x''y'}{((x')^2 + (y')^2)^{3/2}},$$

where we omitted the argument u of the functions κ, x', x'', y' and y''.

Example. *The ellipse.* Let $\boldsymbol{\alpha}\colon \mathbb{R} \longrightarrow \mathbb{R}^2$, $\boldsymbol{\alpha}(u) = (a\cos u, b\sin u)$ for some constants $a, b > 0$. The curve is regular,

$$\kappa(u) = \frac{ab}{(a^2\sin^2 u + b^2\cos^2 u)^{3/2}}.$$

In particular, the curvature is always positive ($\kappa(u) > 0$ for all $u \in \mathbb{R}$), but not constant if $a \neq b$.

Definition 3.5. Let $\boldsymbol{\alpha}\colon I \longrightarrow \mathbb{R}^2$ be a plane regular curve.

(a) A point $\boldsymbol{\alpha}(u_0)$ is an *inflection point* of $\boldsymbol{\alpha}$ if $\kappa(u) = 0$.

(b) A point $\boldsymbol{\alpha}(u_0)$ is a *vertex* of $\boldsymbol{\alpha}$ if $\kappa'(u) = 0$.

Remark. A vertex is well-defined, i.e. the definition does not depend on the parameter.

Example 3.6. (a) *The cubic.* $\alpha(u) = (u, u^3)$. The only inflection point is $\boldsymbol{\alpha}(0) = (0, 0)$, there are no vertices.

(b) *The parabola.* $\boldsymbol{\alpha}(u) = (u, u^2)$. There are no inflection points, the only vertex is at $u = 0$.

(c) *The ellipse.* There are no inflection points, 4 vertices at $u = k\pi/2$.

Theorem 3.7 (The 4-vertex theorem)**.** Any simple smooth regular closed curve has at least 4 vertices.

Here *simple* means the curve has no self-intersections.

Theorem 3.8 (The fundamental theorem of local theory of plane curves)**.** Given a smooth function $\kappa\colon I \longrightarrow \mathbb{R}$, $s_0 \in I$, $a \in \mathbb{R}^2$ and a unit vector $v_0 \in \mathbb{R}^2$, there is a unique smooth regular curve $\boldsymbol{\alpha}\colon I \longrightarrow \mathbb{R}^2$ parametrized by arc length with curvature $\kappa(s)$ and $\boldsymbol{\alpha}(s_0) = a$, $\boldsymbol{\alpha}'(s_0) = v_0$.

3.2 Evolute and involute of a plane curve

Definition 3.9. Let $\boldsymbol{\alpha}\colon I \longrightarrow \mathbb{R}^2$ be a smooth regular curve parametrized by arc length.

(a) Suppose $\kappa(s) \neq 0$, then

$$\rho(s) = \frac{1}{|\kappa(s)|}$$

is called the *radius of curvature*. The point

$$\boldsymbol{e}(s) = \boldsymbol{\alpha}(s) + \frac{1}{\kappa(s)}\boldsymbol{n}(s)$$

is called the *center of curvature*. Here, $\boldsymbol{n}$ is the unit normal of $\boldsymbol{\alpha}$.

(b) The *evolute (caustic)* of the curve $\boldsymbol{\alpha}$ is the curve traced by the centers of curvature. Thus, a parametrization of the evolutive is

$$\boldsymbol{e}\colon I \longrightarrow \mathbb{R}^2, \qquad \boldsymbol{e}(s) = \boldsymbol{\alpha}(s) + \frac{1}{\kappa(s)}\boldsymbol{n}(s).$$

(c) The *involute* of a plane curve $\boldsymbol{\beta}$ is a curve whose evolute is the initial curve $\boldsymbol{\beta}$.

Remark. Properties of the evolute.
$\boldsymbol{\alpha}$, $\boldsymbol{n}$ and κ are smooth, so $\boldsymbol{e}$ is a smooth curve (whenever $\kappa(s) \neq 0$). Moreover,

$$\boldsymbol{e}'(s) = \boldsymbol{\alpha}'(s) + \frac{1}{\kappa(s)}\boldsymbol{n}'(s) - \frac{\kappa'(s)}{\kappa(s)^2}\boldsymbol{n}(s),$$

which implies

$$\boldsymbol{e}'(s) = -\frac{\kappa'(s)}{\kappa(s)^2}\boldsymbol{n}(s).$$

In particular, we have the following conclutions:

(a) $\boldsymbol{e}'(s)$ is *parallel* to the normal vector $\boldsymbol{n}(s)$ of the original curve $\boldsymbol{\alpha}$.

(b) $\boldsymbol{e}'(s) = \boldsymbol{0}$ iff $\kappa'(s) = 0$, i.e., the evolute is *singular* at $\boldsymbol{e}(s_0)$ iff $\boldsymbol{\alpha}(s_0)$ is a *vertex*.

(c) The parameter s is *not* an arc length parameter of the evolute $\boldsymbol{e}$: $\|\boldsymbol{e}'(s)\| = |\frac{\kappa'(s)}{\kappa(s)^2}|$ which is not necessarily 1.

Example 3.10. (a) *The ellipse.* $\boldsymbol{\alpha}(u) = (a\cos u, b\sin u)$ for $a > 0$, $b > 0$ and $a \neq b$.

$$\boldsymbol{e}(u) = (a\cos u, b\sin u) + \frac{a^2\sin^2 u + b^2\cos^2 u}{ab}(-b\cos u, -a\sin u).$$

(b) *The circle.* $\boldsymbol{e}(u) =$ the center.

4 Space curves (curves in $\mathbb{R}^3$)

4.1 The Serret – Frenet formulae

Let $\boldsymbol{\alpha}\colon I \longrightarrow \mathbb{R}^3$ be a smooth regular curve in $\mathbb{R}^3$ parametrized by arc length (i.e., $\boldsymbol{t} = \boldsymbol{\alpha}'$ is the unit tangent vector).

Definition 4.1. The *curvature* $\kappa\colon I \longrightarrow [0, \infty)$ of a space curve $\boldsymbol{\alpha}\colon I \longrightarrow \mathbb{R}^3$ is defined by

$$\kappa(s) := \|\boldsymbol{t}'(s)\|.$$

Remark. The curvature of a *space* curve is always non-negative ($\kappa(s) \geq 0$). For *plane* curves, we introduced the *signed* curvature, which can have negative values. We will see the relation between both concepts later on.

Definition 4.2. Assume that $\kappa(s) > 0$. We define the *principal normal vector* $\boldsymbol{n}(s)$ by

$$\boldsymbol{n}(s) := \frac{1}{\kappa(s)}\boldsymbol{t}'(s).$$

Note that $\boldsymbol{n}(s)$ is really a *unit* vector (and also orthogonal to $\boldsymbol{t}(s)$. We have

$$\boldsymbol{t}'(s) = \kappa(s)\boldsymbol{n}(s).$$

Remark. The *vector product* (or *cross-product*) $\times\colon \mathbb{R}^3 \times \mathbb{R}^3 \longrightarrow \mathbb{R}^3$ in $\mathbb{R}^3$. Recall some facts about the vector product in $\mathbb{R}^3$. Let $\boldsymbol{a}$, $\boldsymbol{b} \in \mathbb{R}^3$.

(a) The *vector product* is defined by

$$\boldsymbol{a} \times \boldsymbol{b} = \begin{pmatrix} a_1 \\ a_2 \\ a_3 \end{pmatrix} \times \begin{pmatrix} a_1 \\ a_2 \\ a_3 \end{pmatrix} = \begin{pmatrix} a_2b_3 - a_3b_2 \\ a_3b_1 - a_1b_3 \\ a_1b_2 - a_2b_1 \end{pmatrix}$$

(b) $\boldsymbol{a} \times \boldsymbol{b}$ is orthogonal to $\boldsymbol{a}$ and $\boldsymbol{b}$, e.g., $(\boldsymbol{a} \times \boldsymbol{b}) \cdot \boldsymbol{a} = 0$.

(c) Antisymmetry: $\boldsymbol{a} \times \boldsymbol{b} = -\boldsymbol{b} \times \boldsymbol{a}$ (in particular, $\boldsymbol{a} \times \boldsymbol{a} = \boldsymbol{0}$).

(d) If $\boldsymbol{a}$ and $\boldsymbol{b}$ are orthogonal unit vectors, then $(\boldsymbol{a}, \boldsymbol{b}, \boldsymbol{a} \times \boldsymbol{b})$ form an orthonormal basis, which is *positively* oriented. Moreover, one has

$$\boldsymbol{b} \times (\boldsymbol{a} \times \boldsymbol{b}) = \boldsymbol{a}, \qquad (\boldsymbol{a} \times \boldsymbol{b}) \times \boldsymbol{a} = \boldsymbol{b}$$

Definition 4.3. The vector $\boldsymbol{b} := \boldsymbol{t} \times \boldsymbol{n}$ is called the *binormal vector* of $\boldsymbol{\alpha}$, and $(\boldsymbol{t}, \boldsymbol{n}, \boldsymbol{b})$ form an orthonormal basis called also *orthonormal frame.*

Since $\boldsymbol{b}'$ is orthogonal to $\boldsymbol{b}$ *and* to $\boldsymbol{t}$, $\boldsymbol{b}'$ is *parallel* to $\boldsymbol{n}$. In particular, the following definition makes sense:

Definition 4.4. The *torsion* $\tau\colon I \longrightarrow \mathbb{R}$ of the space curve $\boldsymbol{\alpha}\colon I \longrightarrow \mathbb{R}^3$ is defined by

$$\boldsymbol{b}'(s) = \tau(s)\boldsymbol{n}(s).$$

Remark. Note that the torsion can have positive or negative values. Moreover, in some books, you will find the equation $\boldsymbol{b}' = -\tau\boldsymbol{n}$ as a definition of the torsion.

Proposition 4.5 (*Serret-Frenet equations*)**.** Let $\boldsymbol{\alpha}\colon I \longrightarrow \mathbb{R}^3$ be a space curve parametrized by arc length with unit tangent, normal and binormal vectors $\boldsymbol{t}$, $\boldsymbol{n}$, $\boldsymbol{b}$. Then

$$\boldsymbol{t}' = \kappa\boldsymbol{n} \tag{4.2}$$

$$\boldsymbol{n}' = -\kappa\boldsymbol{t} - \tau\boldsymbol{b} \tag{4.6}$$

$$\boldsymbol{b}' = \tau\boldsymbol{n} \tag{4.5}$$

or in matrix form

$$\begin{pmatrix} \boldsymbol{t}' \\ \boldsymbol{n}' \\ \boldsymbol{b}' \end{pmatrix} = \begin{pmatrix} 0 & \kappa & 0 \\ -\kappa & 0 & -\tau \\ 0 & \tau & 0 \end{pmatrix} \begin{pmatrix} \boldsymbol{t} \\ \boldsymbol{n} \\ \boldsymbol{b} \end{pmatrix}.$$

Let us now show how to calculate the torsion and curvature for a space curve which is not necessarily parametrized by arc length. This is of practical relevance, since a parametrization is in general not unit speed (i.e., the parameter is not arc length).

Theorem 4.6. Let $\boldsymbol{\alpha}\colon I \longrightarrow \mathbb{R}^3$ be a regular space curve, not necessarily parametrized by arc length. Then the curvature and torsion of $\boldsymbol{\alpha}$ are given by

$$\kappa = \frac{\|\boldsymbol{\alpha}' \times \boldsymbol{\alpha}''\|}{\|\boldsymbol{\alpha}'\|^3} \quad \text{and} \quad \tau = -\frac{(\boldsymbol{\alpha}' \times \boldsymbol{\alpha}'') \cdot \boldsymbol{\alpha}'''}{\|\boldsymbol{\alpha}' \times \boldsymbol{\alpha}''\|^2}$$

(as functions of u), respectively.

Example 4.7. *The helix.* Let $\boldsymbol{\alpha}\colon \mathbb{R} \longrightarrow \mathbb{R}^3$ be given by $\boldsymbol{\alpha}(u) = (a\cos u, a\sin u, u)$ for $a > 0$ (this is a particular case of a helix, see Exercise 4.5). Then $\kappa = \dfrac{a}{a^2+1}$, $\tau(u) = -\dfrac{1}{a^2+1}$.

Remark (Geometric meaning of torsion)**.** The plane through $\boldsymbol{\alpha}(s)$ spanned by $\boldsymbol{t}(s)$ and $\boldsymbol{n}(s)$ is called the *osculating plane.*

The torsion of a curve measures the rate at which the curve pulls away from the osculating plane.

Proposition 4.8 (Exercise)**.** Let $\boldsymbol{\alpha} : I \to \mathbb{R}^3$ be a smooth curve, $\boldsymbol{\alpha}' \times \boldsymbol{\alpha}'' \neq \boldsymbol{0}$ for $u \in I$. Assume that there is a plane $\Pi \subset \mathbb{R}^3$ containing $\boldsymbol{\alpha}(I)$. Then $\tau(u) \equiv 0$.

We can now express one of the main results on space curve (similar to Theorem 3.8):

Theorem 4.9 (The fundamental theorem of local theory of space curves)**.** Given smooth functions $\kappa\colon I \longrightarrow (0,\infty)$ and $\tau\colon I \longrightarrow \mathbb{R}$, there exists a smooth regular curve $\boldsymbol{\alpha}\colon I \longrightarrow \mathbb{R}^3$ parametrized by arc length such that κ and τ are the curvature and torsion of $\boldsymbol{\alpha}$. Moreover, $\boldsymbol{\alpha}$ is unique up to translations (of the *starting point*) and rotation (of the *starting orthonormal basis*).

Remark 4.10. *Local canonical form of a space curve.* Let $\boldsymbol{\alpha}\colon I \longrightarrow \mathbb{R}^3$ be a space curve parametrized by arc length with $0 \in I$. Then

$$\begin{aligned}\boldsymbol{\alpha}(s) &= \boldsymbol{\alpha}(0) + s\boldsymbol{\alpha}'(0) + \frac{s^2}{2!}\boldsymbol{\alpha}''(0) + \frac{s^3}{3!}\boldsymbol{\alpha}'''(0) + O(s^4)\\ &= \boldsymbol{\alpha}(0) + s\boldsymbol{t}(0) + \frac{s^2}{2!}\underbrace{\boldsymbol{t}'(0)}_{=\kappa(0)\boldsymbol{n}(0)} + \frac{s^3}{3!}\underbrace{\boldsymbol{t}''(0)}_{=\kappa'(0)\boldsymbol{n}(0)+\kappa(0)(-\kappa(0)\boldsymbol{t}(0)-\tau(0)\boldsymbol{b}(0))} + O(s^4)\end{aligned}$$

by the Serret-Frenet formulae. In paricular,

$$\boldsymbol{\alpha}(s) - \boldsymbol{\alpha}(0) = \Big(s - \frac{\kappa(0)^2 s^3}{6}\Big)\boldsymbol{t}(0) + \Big(\frac{\kappa(0)s^2}{2} + \frac{\kappa'(0)s^3}{6}\Big)\boldsymbol{n}(0) - \frac{\kappa(0)\tau(0)s^3}{6}\boldsymbol{b}(0) + O(s^4).$$

If we choose the coordinate system such that $\boldsymbol{t}(0) = (1,0,0)$, $\boldsymbol{n}(0) = (0,1,0)$ and $\boldsymbol{b}(0) = (0,0,1)$, and if we write $\boldsymbol{\alpha}(s) - \boldsymbol{\alpha}(0) = (x(s), y(s), z(s))$, then

$$\begin{aligned}x(s) &= s - \frac{\kappa(0)^2 s^3}{6}\\ y(s) &= \frac{\kappa(0)s^2}{2} + \frac{\kappa'(0)s^3}{6}\\ z(s) &= -\frac{\kappa(0)\tau(0)s^3}{6}.\end{aligned}$$

These equations are called the *local canonical form* of a space curve $\boldsymbol{\alpha}$.

5 A bit of Analysis (should have been a reminder)

We consider the Euclidean space

$$\mathbb{R}^n = \big\{\, \boldsymbol{x} = (x_1, \dots, x_n) \,\big|\, x_i \in \mathbb{R}, i = 1, \dots, n \,\big\}$$

Definition 5.1.

(a) A *ball of radius $r > 0$ with center $\boldsymbol{a} \in \mathbb{R}^n$* in $\mathbb{R}^n$ is defined by

$$B_r(\boldsymbol{a}) := \big\{\, \boldsymbol{x} \in \mathbb{R}^n \,\big|\, \|\boldsymbol{x} - \boldsymbol{a}\| = \sqrt{(x_1 - a_1)^2 + \ldots + (x_n - a_n)^2} < r \,\big\}.$$

(b) A subset $U \subset \mathbb{R}^n$ is called *open*, if for any $\boldsymbol{y} \in U$ there exists $r > 0$ such that $B_r(\boldsymbol{y}) \subset U$, i.e.

$$\forall\, \boldsymbol{y} \in U\ \exists\, r > 0 : \quad B_r(\boldsymbol{y}) \subset U.$$

Example 5.2.

(a) Interval $(a, b) \subset \mathbb{R}$ is open.

(b) Closed interval $[a, b] \subset \mathbb{R}$ is not open.

(c) The ball $B_r(\boldsymbol{a})$ is an open subset of $\mathbb{R}^n$ for any $\boldsymbol{a} \in \mathbb{R}^n$ and $r > 0$.

(d) The *(open) cube* $(a_1, b_1) \times \ldots \times (a_n, b_n)$ is an open subset for any a_i, $b_i \in \mathbb{R}$ with $a_i < b_i$. Note that for $n = 1$, a cube is an interval, and for $n = 2$, a cube is a rectangle (without the boundary).

(e) The entire space $\mathbb{R}^n$ and the empty set $\emptyset$ are open.

Now let $U \subset \mathbb{R}^n$ be open, $\boldsymbol{f}\colon U \longrightarrow \mathbb{R}^m$ be a map, i.e.,

$$\boldsymbol{f}(\boldsymbol{u}) = \begin{pmatrix} f_1(u_1, \dots, u_n) \\ \vdots \\ f_m(u_1, \dots, u_n) \end{pmatrix}$$

for any $\boldsymbol{u} = (u_1, \dots, u_n) \in U$. We say that $\boldsymbol{f}$ is *smooth* if the (scalar) functions $f_i\colon U \longrightarrow \mathbb{R}$ are smooth for all $i = 1, \dots m$, i.e., if all partial derivatives of all order exist and are continuous.

Example 5.3.

(a) $\boldsymbol{f}\colon \mathbb{R}^2 \longrightarrow \mathbb{R}^3$ ($U = \mathbb{R}^2$, $n = 2$, $m = 3$) with

$$\boldsymbol{f}(u_1, u_2) = \begin{pmatrix} u_1 \\ u_2 \\ u_1^2 + u_2^2 \end{pmatrix}$$

is a smooth map.

(b) $\boldsymbol{f}\colon B_1(\mathbf{0}) \longrightarrow \mathbb{R}^3$ ($U = B_1(\mathbf{0}) \subset \mathbb{R}^2$, $n = 2$, $m = 3$) with

$$\boldsymbol{f}(u_1, u_2) = \begin{pmatrix} u_1 \\ u_2 \\ \sqrt{1 - u_1^2 - u_2^2} \end{pmatrix}$$

is a smooth map as well.

For (scalar) functions, even of more than one variable, we know how to derive, e.g., if $f(x, y) = x^2 y + 3y^3$, then

$$\frac{\partial f}{\partial x}(x, y) = 2xy \quad \text{and} \quad \frac{\partial f}{\partial y}(x, y) = x^2 + 9y^2.$$

Definition 5.4. Let $U \subset \mathbb{R}^n$ be open, let $\boldsymbol{f}\colon U \longrightarrow \mathbb{R}^m$ be a smooth map and let $\boldsymbol{p} \in U$. The *Jacobi matrix* of $\boldsymbol{f}$ at $\boldsymbol{p}$ is the $(m \times n)$-matrix given by

$$J_{\boldsymbol{p}}\boldsymbol{f} := \begin{pmatrix} \partial_1 f_1(\boldsymbol{p}) & \dots & \partial_n f_1(\boldsymbol{p}) \\ \vdots & & \vdots \\ \partial_1 f_m(\boldsymbol{p}) & \dots & \partial_n f_m(\boldsymbol{p}) \end{pmatrix} \qquad \text{where} \quad \partial_i f_j(\boldsymbol{p}) := \frac{\partial}{\partial u_i} f_j(u)\bigg|_{u=p}, \quad i = 1, \dots, n.$$

The *derivative* of $\boldsymbol{f}$ at p is the linear map

$$\mathrm{d}_{\boldsymbol{p}}\boldsymbol{f}\colon \mathbb{R}^n \longrightarrow \mathbb{R}^m, \quad h \mapsto (\mathrm{d}_{\boldsymbol{p}}\boldsymbol{f})(h) = J_{\boldsymbol{p}}\boldsymbol{f} \cdot h$$

Note that the Jacobi matrix is just the matrix representation of the derivative in the standard basis.

Remark. Since $\mathrm{d}_{\boldsymbol{p}}\boldsymbol{f}$ is linear, its image (range) $(\mathrm{d}_{\boldsymbol{p}}\boldsymbol{f})(\mathbb{R}^n)$ is a vector subspace of $\mathbb{R}^m$, spanned by

$$\{(\mathrm{d}_{\boldsymbol{p}}\boldsymbol{f})(\boldsymbol{e}_1), \dots, (\mathrm{d}_{\boldsymbol{p}}\boldsymbol{f})(\boldsymbol{e}_n)\},$$

where $\{\boldsymbol{e}_1, \dots, \boldsymbol{e}_n\}$ is the standard basis in $\mathbb{R}^n$. Observe that

$$(\partial_i \boldsymbol{f}(\boldsymbol{p}) :=)(\mathrm{d}_{\boldsymbol{p}}\boldsymbol{f})(\boldsymbol{e}_i) = \begin{pmatrix} \partial_i f_1(\boldsymbol{p}) \\ \vdots \\ \partial_i f_m(\boldsymbol{p}) \end{pmatrix}$$

which is just the i^{th} column of the Jacobi matrix $J_{\boldsymbol{p}}\boldsymbol{f}$.

Example 5.5.

(a) $\boldsymbol{f}\colon \mathbb{R}^2 \longrightarrow \mathbb{R}^3$

$$\boldsymbol{f}(u,v) = \begin{pmatrix} u \\ v \\ u^2+v^2 \end{pmatrix} \quad \text{then} \quad J_{(u,v)}\boldsymbol{f} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \\ 2u & 2v \end{pmatrix}.$$

At $\boldsymbol{p} = (0,0)$, the image of $\mathrm{d}_{\boldsymbol{p}}\boldsymbol{f}$ is spanned by $(1,0,0)$ and $(0,1,0)$.

(b) $\boldsymbol{f}\colon \mathbb{R}^2 \longrightarrow \mathbb{R}^3$,

$$\boldsymbol{f}(u,v) = \begin{pmatrix} u \\ v^2 \\ uv \end{pmatrix} \quad \text{then} \quad J_{(u,v)}\boldsymbol{f} = \begin{pmatrix} 1 & 0 \\ 0 & 2v \\ v & u \end{pmatrix}.$$

At $\boldsymbol{p} = (0,0)$, the image of $\mathrm{d}_{\boldsymbol{p}}\boldsymbol{f}$ is spanned by $\{(1,0,0),(0,0,0)\}$, i.e., by $(1,0,0)$ (the x-axis).

(c) $f\colon \mathbb{R}^3 \longrightarrow \mathbb{R}$,

$$f(x,y,z) := 2x^2 + y^2 - z^2, \qquad J_{(x,y,z)}f = (4x, 2y, -2z)$$

(the *gradient* of f). Note that the Jacobi matrix of a scalar function is just the gradient. Here, the image of $\mathrm{d}_{\boldsymbol{p}}f$ is either $\mathbb{R}$ (if $(x,y,z) \neq \mathbf{0}$) or $\{0\}$ (if $(x,y,z) = \mathbf{0}$).

Let us finally motivate the *implicit function theorem*

Example 5.6. Let $f\colon \mathbb{R}^2 \longrightarrow \mathbb{R}$ be given by $f(u,v) = u^2 + v^2$. We want to solve the equation

$$f(u,v) = c$$

near some point $(a,b) \in \mathbb{R}^2$ for $c := f(a,b) \geq 0$, i.e., we look for a function $g(u) = v$ such that $f(u, g(u)) = c$. The implicit function tells us that if $\partial_v f(u_0, v_0) \neq 0$ then this is possible. Here, $\partial_v f(a,b) = 2b$, and a simple calculation shows that

$$f(u,v) = c \iff v = \begin{cases} \sqrt{c-u^2}, & \text{if } b > 0, \\ -\sqrt{c-u^2}, & \text{if } b < 0. \end{cases}$$

Theorem 5.7 (Implicit function theorem)**.** Let $W \subset \mathbb{R}^p \times \mathbb{R}^m$ be open and $\boldsymbol{f}\colon W \longrightarrow \mathbb{R}^m$ be smooth. Let $(\boldsymbol{a},\boldsymbol{b}) \in W$ ($\boldsymbol{a} \in \mathbb{R}^p$, $\boldsymbol{b} \in \mathbb{R}^m$) and $\boldsymbol{c} := f(\boldsymbol{a},\boldsymbol{b}) \in \mathbb{R}^m$. Consider a function $\boldsymbol{\varphi} : W \cap \mathbb{R}^m \to \mathbb{R}^m$ defined by $\boldsymbol{y} \mapsto \boldsymbol{f}(\boldsymbol{a},\boldsymbol{y})$. Its Jacobi matrix is

$$J(\boldsymbol{a},\boldsymbol{y}) = \frac{\partial \boldsymbol{f}}{\partial \boldsymbol{y}}(\boldsymbol{a},\boldsymbol{y}) = \begin{pmatrix} \dfrac{\partial f_1}{\partial y_1}(\boldsymbol{a},\boldsymbol{y}) & \dots & \dfrac{\partial f_1}{\partial y_m}(\boldsymbol{a},\boldsymbol{y}) \\ \vdots & & \vdots \\ \dfrac{\partial f_m}{\partial y_1}(\boldsymbol{a},\boldsymbol{y}) & \dots & \dfrac{\partial f_m}{\partial y_m}(\boldsymbol{a},\boldsymbol{y}) \end{pmatrix}$$

Assume that $J(\boldsymbol{a},\boldsymbol{y})$ is invertible at $\boldsymbol{y} = \boldsymbol{b}$. Then there exist open sets $U \subset \mathbb{R}^p$, $\boldsymbol{a} \in U$, and $V \subset \mathbb{R}^m$, $\boldsymbol{b} \in V$, and a smooth map $\boldsymbol{g} : U \to V$ with $\boldsymbol{g}(\boldsymbol{a}) = \boldsymbol{b}$ such that

$$\{(\boldsymbol{x},\boldsymbol{y}) \in U \times V \mid \boldsymbol{f}(\boldsymbol{x},\boldsymbol{y}) = \boldsymbol{c}\} = \{(\boldsymbol{x}, \boldsymbol{g}(\boldsymbol{x})) \mid \boldsymbol{x} \in U\}$$

(i.e. the level set of points $(\boldsymbol{x},\boldsymbol{y})$ with $\boldsymbol{f}(\boldsymbol{x},\boldsymbol{y}) = \boldsymbol{c}$ is locally a *graph* of some smooth function $\boldsymbol{g} : U \to V$).

We will use this theorem in a particular case of $m = 1$: having a function

$$f : \mathbb{R}^{p+1} \to \mathbb{R}, \quad (x_1, \dots, x_p, y) \mapsto f(\boldsymbol{x}, y), \quad f(\boldsymbol{x}_0, y_0) = c$$

with $\dfrac{\partial f}{\partial y}(\boldsymbol{x}_0, y_0) \neq 0$, one has $y = g(\boldsymbol{x})$ in a neighborhood of $\boldsymbol{x}_0$ for $f(\boldsymbol{x}, y) = c$.

6 Surfaces

Recall that we defined a curve as a smooth map $\boldsymbol{\alpha}\colon I \longrightarrow \mathbb{R}^n$. So a curve is a deformation of an interval, i.e., a piece of the real line.

Similarly, we look to define a surface as a deformation of an open subset in $\mathbb{R}^2$. Intuitively, a surface in $\mathbb{R}^n$ ($n \geq 3$) is a subset of $\mathbb{R}^n$ that looks locally like a subset of $\mathbb{R}^2$.

6.1 Parametrizations of regular surfaces

Definition 6.1. A subset $S \subset \mathbb{R}^3$ is a *regular surface* if for every point $p \in S$ there exists an open set V in $\mathbb{R}^3$ containing p and a map $\boldsymbol{x}\colon U \longrightarrow S \cap V$, where U is an open subset of $\mathbb{R}^2$, such that

(a) $\boldsymbol{x}$ is a smooth map; that is, if

$$\boldsymbol{x}(u, v) = (x_1(u, v), x_2(u, v), x_3(u, v))$$

then x_1, x_2, x_3 are smooth functions.

(b) $\boldsymbol{x}\colon U \longrightarrow S \cap V$ is a homeomorphism, that is, $\boldsymbol{x}$ has a continuous inverse $\boldsymbol{x}^{-1}\colon S \cap V \longrightarrow U$ *(this condition excludes self-intersections).*

(c) The partial derivatives $\boldsymbol{x}_u$ and $\boldsymbol{x}_v$ are linearly independent for all $(u, v) \in U$ *(this condition excludes singularities and dimension reduction).*

$\boldsymbol{x}$ is called a *local parametrization* of S at p, and $\boldsymbol{x}^{-1}$ is called a *local coordinate chart.*

Let us now come to some main classes of examples of surfaces:

6.2 Graphs of functions and level sets as surfaces

Proposition 6.2. Let $U \subset \mathbb{R}^2$ be open and $g\colon U \longrightarrow \mathbb{R}$ be a smooth function. Then the graph of g,

$$\operatorname{graph}(g) := \left\{ (u, v, g(u, v) \in \mathbb{R}^3 \,\middle|\, (u, v) \in U \right\}$$

is a regular surface in $\mathbb{R}^3$.

Example 6.3.

(a) Let $U = \mathbb{R}^2$ and

$$g(u, v) = \frac{u^2}{a^2} + \frac{v^2}{b^2},$$

then the graph of g is a surface: an *elliptic paraboloid.*

(b) Similarly, let

$$g(u, v) = \frac{u^2}{a^2} - \frac{v^2}{b^2},$$

then the graph of g is a *hyperbolic paraboloid.*

Example 6.4. The *sphere* of radius $r > 0$ and center $\mathbf{0}$ is defined as

$$S(r) := \left\{ (x, y, z) \in \mathbb{R}^3 \,\middle|\, x^2 + y^2 + z^2 - r^2 = 0 \right\}.$$

Example 6.5. Consider the function $f\colon \mathbb{R}^3 \longrightarrow \mathbb{R}$ given by $f(x,y,z) = x^2 + y^2 + z^2$. Then the sphere $S(r)$ of radius $r > 0$ is the level set r^2 of f, i.e.,

$$S(r) = \left\{ (x,y,z) \in \mathbb{R}^3 \,\middle|\, f(x,y,z) = r^2 \right\} =: f^{-1}(r^2)$$

All the level sets $f^{-1}(r^2)$ are *regular surfaces*, except for $c = r^2 = 0$. The value $c = 0$ corresponds to the point $\boldsymbol{x} = (x,y,z) = \mathbf{0}$. Note that

$$\nabla f = (\partial_x f, \partial_y f, \partial_z) = (2x, 2y, 2z)$$

and that $\nabla f(\boldsymbol{x}) = 0$ iff $\boldsymbol{x} = \mathbf{0}$. We have to exclude such values!

Definition 6.6. Let $U \subset \mathbb{R}^3$ be open and $f\colon U \longrightarrow \mathbb{R}$ be smooth. A value $c \in \mathbb{R}$ in the range $f(U)$ of f is called *regular value* of f if $\nabla f(\boldsymbol{p}) = (\partial_x f, \partial_y f, \partial_z f)(\boldsymbol{p}) \neq \mathbf{0}$ for all $\boldsymbol{p} \in U$ such that $f(\boldsymbol{p}) = c$.

A point $\boldsymbol{p}$ is called *critical point* if $\nabla f(\boldsymbol{p}) = \mathbf{0}$. In this case $c = f(\boldsymbol{p})$ is a *critical value* of f.

So $c = r^2 > 0$ is a regular value of f from the previous example, and $c = 0$ is a critical value.

Proposition 6.7. Let $U \subset \mathbb{R}^3$ be open and $f\colon U \longrightarrow \mathbb{R}$ be smooth, let $c \in f(U)$ be a regular value of f. Then

$$f^{-1}(c) := \left\{ \boldsymbol{x} \in U \,\middle|\, f(\boldsymbol{x}) = c \right\}$$

is a regular surface.

Example 6.8.

(a) $S(r) = \{ (x,y,z) \in \mathbb{R}^3 \,|\, x^2 + y^2 + z^2 = r^2 \}$ is the level set of f, where $f(x,y,z) = x^2 + y^2 + z^2$, i.e., $S(r) = f^{-1}(r^2)$. $S(r)$ is a regular surface if $r > 0$.

(b) Let $f\colon \mathbb{R}^3 \longrightarrow \mathbb{R}$ be given by $f(x,y,z) = x^2 + y^2 - z^2$. Let $S = f^{-1}(1)$ be the level set 1 of f. Since $c = 1$ is a regular value of f, S is a regular surface, a *hyperboloid of one sheet*.

(c) With the same f as before, $f^{-1}(-1)$ is called the *hyperboloid of two sheets*. The value -1 is again a regular value, so the hyperboloid of two sheets is regular.

(d) A cylinder given by those points $(x,y,z) \in \mathbb{R}^3$ such that $x^2 + y^2 = 1$ is a regular surface.

6.3 Change of parameters

Definition 6.9. Let U, V be two open sets. A smooth map $\boldsymbol{h}\colon V \longrightarrow U$ is called a *diffeomorphism* if it is bijective and if the inverse $\boldsymbol{h}^{-1}\colon U \longrightarrow V$ is also smooth.

Example 6.10. Let $U = V = \mathbb{R}$. Then $\boldsymbol{h}(x) = x$ is a diffeomorphism, but $\boldsymbol{h}(x) = x^3$ is not.

Proposition 6.11. (a) Let $S \subset \mathbb{R}^3$ be a surface and let $\boldsymbol{x}\colon U \subset \mathbb{R}^2 \longrightarrow S$ be a local parametrization. Let $\boldsymbol{h}\colon V \subset \mathbb{R}^2 \longrightarrow U$ be a diffeomorphism. Then $\boldsymbol{y} = \boldsymbol{x} \circ \boldsymbol{h}\colon V \longrightarrow S$ is also a local parametrization.

(b) Let $\boldsymbol{x}\colon U \longrightarrow S$ and $\boldsymbol{y}\colon V \longrightarrow S$ be two local parametrizations with $\boldsymbol{x}(U) = \boldsymbol{y}(V) \subset S$ (i.e., $\boldsymbol{x}$ and $\boldsymbol{y}$ cover the same region of the surface). Then $\boldsymbol{x}^{-1} \circ \boldsymbol{y}\colon V \longrightarrow U$ is a diffeomorphism.

6.4 Special surfaces

Surfaces constructed by a plane and space curves.

Example 6.12. Surface of revolution. Let I be an open interval in $\mathbb{R}$ and $\tilde{\boldsymbol{\alpha}}\colon I \longrightarrow \mathbb{R}^2$ be a regular smooth plane curve, $\tilde{\boldsymbol{\alpha}}(v) = (f(v), g(v))$. Define a space curve $\boldsymbol{\alpha}(v) = (f(v), 0, g(v))$. Assume that $\boldsymbol{\alpha}$ has no self-intersections (i.e. $\boldsymbol{\alpha}(u) \neq \boldsymbol{\alpha}(v)$ if $u \neq v$) and that $f(v) \neq 0$, so $\boldsymbol{\alpha}$ does not meet the z-axis.

Now rotate $\boldsymbol{\alpha}$ about the z-axis. The set

$$S := \left\{ (f(v)\cos u, f(v)\sin u, g(v)) \,\middle|\, u \in \mathbb{R}, v \in I \right\}$$

is a surface, called a *surface of revolution.*
The curve $\boldsymbol{\alpha}$ is called the *generating curve.* The circles swept out by points of $bm\alpha$ are called *parallels*, and the curves obtained by rotating $\boldsymbol{\alpha}$ through a fixed angle are *meridians.*

Examples: cylinder ($\boldsymbol{\alpha}$ is a vertical line), *catenoid* ($\boldsymbol{\alpha}(v) = (\cosh v, 0, v)$, $v \in \mathbb{R}$).

Example 6.13. Canal surfaces.
Let $\boldsymbol{\alpha}\colon I \longrightarrow \mathbb{R}^3$ be a smooth regular non-self-intersecting space curve parametrized by arc length. Choose $r > 0$ small enough, and consider the family of circles in the normal plane (i.e., spanned by $\boldsymbol{n}(s)$ and $\boldsymbol{b}(s)$ with center $\boldsymbol{\alpha}(s)$ and radius r. These form a surface called a *canal surface* or *tubular neighbourhood* of $\boldsymbol{\alpha}$. This surface is parametrized by

$$\boldsymbol{x}(s, \vartheta) = \boldsymbol{\alpha}(s) + r(\boldsymbol{n}(s)\cos\vartheta + \boldsymbol{b}(s)\sin\vartheta).$$

Example 6.14. Ruled surfaces. Let $\boldsymbol{\alpha}\colon I \longrightarrow \mathbb{R}^3$ be a smooth regular space curve (without self-intersections) and $\boldsymbol{w}\colon I \longrightarrow \mathbb{R}^3$ be a smooth map which is never zero. Suppose that $\boldsymbol{\alpha}'(u)$ is not parallel to $\boldsymbol{w}(u)$ (where $\boldsymbol{w}(u)$ is viewed as a vector). We consider the family of segments of lines through $\boldsymbol{\alpha}(u)$ and parallel to $\boldsymbol{w}(u)$.

These form a surface call a *ruled surface.* If we take $J = (-a, a)$, with a small enough, then

$$\boldsymbol{x}(u, v) = \boldsymbol{\alpha}(u) + v\boldsymbol{w}(u), u \in I, v \in J$$

is a parametrization of a ruled surface.

Example 6.15. $f(x, y, z) := x^2 + y^2 - z^2$ defines a smooth function $f\colon \mathbb{R}^3 \longrightarrow \mathbb{R}$, and 1 is a regular value, hence $S = f^{-1} = \{ (x, y, z) \,|\, x^2 + y^2 - z^2 = 1 \}$ is a regular surface, a *hyperboloid of one sheet.* It is a surface of revolution and a ruled surface.

7 Tangent plane, first fundamental form and area

7.1 The tangent plane

Definition 7.1. Let S be a regular surface and $p \in S$. A *tangent vector* to S at p is the tangent vector $\alpha'(0) \in \mathbb{R}^3$ of a smooth (not necessarily regular) curve $\alpha\colon (-\varepsilon, \varepsilon) \longrightarrow S \subset \mathbb{R}^3$ with $\alpha(0) = p$ (for some $\varepsilon > 0$).

Let $\boldsymbol{x}\colon U \longrightarrow S$ be a local parametrization of S, $\boldsymbol{q} \in U$, $\boldsymbol{x}(\boldsymbol{q}) = \boldsymbol{p}$. Recall that the differential (or derivative) $d_{\boldsymbol{q}}\boldsymbol{x}$ is a linear map $d_{\boldsymbol{q}}\boldsymbol{x} : \mathbb{R}^2 \to \mathbb{R}^3$. By the definition of a regular surface, $d_{\boldsymbol{q}}\boldsymbol{x}$ has full rank at every point, so the dimension of the image is equal to 2.

Definition 7.2. The plane $d_{\boldsymbol{q}}\boldsymbol{x}(\mathbb{R}^2)$ is called the *tangent plane* to S at $\boldsymbol{p}$ and is denoted by $T_{\boldsymbol{p}}S$.

Proposition 7.3. Let $\boldsymbol{x}: U \longrightarrow S$ be a local parametrization of a regular surface S with $U \subset \mathbb{R}^2$ open, and let $\boldsymbol{q} \in U$. Then Then the tangent plane $T_{\boldsymbol{p}}S$ coincides with the set of all tangent vectors to S at $\boldsymbol{p}$.

Remark 7.4. (a) Since the definition of a tangent vector does not depend on a parametrization, Prop. 7.3 implies that the tangent plane does not depend on a parametrization either.

(b) If $\boldsymbol{\alpha}(s) = \boldsymbol{x}(u(s), v(s))$ and $\boldsymbol{w} = \alpha'(0)$, then w has coordinates $(u'(0), v'(0))$ with respect to the basis $\{\boldsymbol{x}_u(\boldsymbol{q}), \boldsymbol{x}_v(\boldsymbol{q})\}$.

Example 7.5.

(a) **Tangent plane to graph of a function:** Let $g: U \longrightarrow \mathbb{R}$ be a smooth function on an open subset U of $\mathbb{R}^2$, i.e.

$$S := \operatorname{graph} g = \{\, (u, v, g(u, v)) \,|\, (u, v) \in U \,\}$$

is a regular surface with parametrisation $\boldsymbol{x}(u, v) := (u, v, g(u, v))$. Then the tangent plane $T_{\boldsymbol{p}}S$ to S at $\boldsymbol{p} = (u, v, g(u, v))$ is generated by

$$\{\boldsymbol{x}_u(\boldsymbol{q}), \boldsymbol{x}_v(\boldsymbol{q})\} = \{(1, 0, g_u(u, v)), (0, 1, g_v(u, v))\},$$

where $\boldsymbol{q} = (u, v)$.

(b) **Tangent plane to a level set of a function:** Let $f: \mathbb{R}^3 \longrightarrow \mathbb{R}$ be a smooth function, and let $c \in \mathbb{R}$ be a regular value of f (i.e., $\nabla f(\boldsymbol{p}) \neq \boldsymbol{0}$ for all $\boldsymbol{p} \in \mathbb{R}^3$ with $f(\boldsymbol{p}) = c$). We have seen that $S := f^{-1}(c)$ is a regular surface.

Lemma 7.6. Let $\boldsymbol{p} \in S$, then $T_{\boldsymbol{p}}S$ is the plane in $\mathbb{R}^3$ *orthogonal* to $\nabla f(\boldsymbol{p})$.

7.2 The first fundamental form

Let $\boldsymbol{p} \in S$. We can consider the restriction of the inner product $(\cdot): \mathbb{R}^3 \times \mathbb{R}^3 \longrightarrow \mathbb{R}$, $(\boldsymbol{v}, \boldsymbol{w}) \mapsto \boldsymbol{v} \cdot \boldsymbol{w}$, to $T_{\boldsymbol{p}}S \subset \mathbb{R}^3$. We denote the restriction by $\langle \cdot, \cdot \rangle_{\boldsymbol{p}}$, i.e.,

$$\langle \cdot, \cdot \rangle_{\boldsymbol{p}}: T_{\boldsymbol{p}}S \times T_{\boldsymbol{p}}S \longrightarrow \mathbb{R}, \qquad (\boldsymbol{w}_1, \boldsymbol{w}_2) \mapsto \boldsymbol{w}_1 \cdot \boldsymbol{w}_2.$$

This map is

- *bilinear*, i.e, linear in both of its arguments;
- *symmetric*, i.e., $\langle \boldsymbol{w}_2, \boldsymbol{w}_1 \rangle_{\boldsymbol{p}} = \langle \boldsymbol{w}_1, \boldsymbol{w}_2 \rangle_{\boldsymbol{p}}$ for all $\boldsymbol{w}_1, \boldsymbol{w}_2 \in T_{\boldsymbol{p}}S$;
- and *positive*, i.e., $\|\boldsymbol{w}\|_{\boldsymbol{p}}^2 := \langle \boldsymbol{w}, \boldsymbol{w} \rangle \geq 0$ and $\|\boldsymbol{w}\|_{\boldsymbol{p}}^2 = 0$ implies $\boldsymbol{w} = 0$ for all $\boldsymbol{w} \in T_{\boldsymbol{p}}S$.

We can now measure the length of a tangent vector $\boldsymbol{w} \in T_{\boldsymbol{p}}S$ and the angle between two tangent vectors $\boldsymbol{w}_1, \boldsymbol{w}_2 \in T_{\boldsymbol{p}}S$ by

$$\sqrt{\langle \boldsymbol{w}, \boldsymbol{w} \rangle_{\boldsymbol{p}}} \qquad \text{and} \qquad \cos \vartheta = \frac{\langle \boldsymbol{w}_1, \boldsymbol{w}_2 \rangle_{\boldsymbol{p}}}{\sqrt{\langle \boldsymbol{w}_1, \boldsymbol{w}_1 \rangle_{\boldsymbol{p}}} \sqrt{\langle \boldsymbol{w}_2, \boldsymbol{w}_2 \rangle_{\boldsymbol{p}}}}.$$

A quadratic form $I_{\boldsymbol{p}}$ is obtained from a bilinear form $\langle \cdot, \cdot \rangle_{\boldsymbol{p}}$ by setting $I_{\boldsymbol{p}}(\boldsymbol{w}) := \langle \boldsymbol{w}, \boldsymbol{w} \rangle_{\boldsymbol{p}}$.

Definition 7.7. The quadratic form $I_{\boldsymbol{p}}: T_{\boldsymbol{p}}S \longrightarrow \mathbb{R}$, $I_{\boldsymbol{p}}(\boldsymbol{w}) := \langle \boldsymbol{w}, \boldsymbol{w} \rangle_{\boldsymbol{p}} = \|\boldsymbol{w}\|^2$ is called the *first fundamental form* at $\boldsymbol{p} \in S$.

Definition 7.8. The functions $E, F, G\colon U \longrightarrow \mathbb{R}$ defined by

$$E := \langle \boldsymbol{x}_u, \boldsymbol{x}_u \rangle_{\boldsymbol{p}}, \quad F := \langle \boldsymbol{x}_u, \boldsymbol{x}_v \rangle_{\boldsymbol{p}}, \quad G := \langle \boldsymbol{x}_v, \boldsymbol{x}_v \rangle_{\boldsymbol{p}}$$

are called the *coefficients* of the first fundamental form in the local parametrization $\boldsymbol{x}\colon U \longrightarrow S$.

Note that the coefficients of the first fundamental form depend on the parametrisation $\boldsymbol{x}$!

Remark 7.9. If $(a, b) \in \mathbb{R}^2$ are the coordinates of a vector $\boldsymbol{w} \in T_{\boldsymbol{p}}S$ with respect to the basis $\{\boldsymbol{x}_u(\boldsymbol{q}), \boldsymbol{x}_v(\boldsymbol{q})\}$, then

$$I_{\boldsymbol{p}}(\boldsymbol{w}) = a^2 E + 2abF + b^2 G = \begin{pmatrix} a & b \end{pmatrix} \cdot \begin{pmatrix} E & F \\ F & G \end{pmatrix} \cdot \begin{pmatrix} a \\ b \end{pmatrix}.$$

Since $I_{\boldsymbol{p}}$ is positive ($I_{\boldsymbol{p}}(\boldsymbol{w}) = \|\boldsymbol{w}\|^2 \geq 0$ and $I_{\boldsymbol{p}}(\boldsymbol{w}) = 0$ implies $\boldsymbol{w} = \boldsymbol{0}$), we have

$$E > 0, \quad G > 0 \quad \text{and} \quad \det \begin{pmatrix} E & F \\ F & G \end{pmatrix} = EG - F^2 > 0.$$

Example 7.10. Let S be a plane in $\mathbb{R}^3$ given by an equation $ax + by + cz + d = 0$, and assume without loos of generality that $c \neq 0$. Then

$$\boldsymbol{x}_x(x, y) = (1, 0, -a/c) \qquad \text{and} \qquad \boldsymbol{x}_y(x, y) = (0, 1, -b/c).$$

In particular, we have

$$E(x, y) = 1 + \frac{a^2}{c^2}, \qquad F(x, y) = \frac{ab}{c^2}, \qquad G(x, y) = 1 + \frac{b^2}{c^2}$$

Example 7.11. Coefficients of the first fundamental form for a graph of a function: Let a surface be given by a graph of a function g, namely $\boldsymbol{x}(u, v) := (u, v, g(u, v)) = (u, v, u^2 + v^2)$ for $(u, v) \in U := \mathbb{R}^2$. Then

$$\boldsymbol{x}_u(u, v) = (1, 0, g_u) = (1, 0, 2u) \qquad \text{and} \qquad \boldsymbol{x}_v(u, v) = (0, 1, g_v) = (0, 1, 2v).$$

In particular, we have

$$\begin{aligned} E &= (1, 0, g_u) \cdot (1, 0, g_u) = 1 + g_u^2, && \text{here} \quad E(u, v) = 1 + 4u^2, \\ F &= (1, 0, g_u) \cdot (0, 1, g_v) = g_u g_v, && \text{here} \quad F(u, v) = 8uv, \\ G &= (0, 1, g_v) \cdot (0, 1, g_v) = 1 + g_v^2 && \text{here} \quad G(u, v) = 1 + 4v^2, \end{aligned}$$

Example 7.12. Coefficients of the first fundamental form for a surface of revolution: Let S be obtained by rotating the space curve given by $\boldsymbol{\alpha}(v) = (f(v), 0, g(v))$, $v \in \mathbb{R}$, around the z-axis (without self-intersections and without meeting the z-axis, i.e., $f(v) = 0$). A parametrization is then given by

$$\boldsymbol{x}(u, v) = (f(v) \cos u, f(v) \sin u, g(v))$$

$(u, v) \in (-\pi, \pi) \times \mathbb{R}$. Here, we have

$$\boldsymbol{x}_u(u, v) = (-f(v) \sin u, f(v) \cos u, 0) \quad \text{and} \quad \boldsymbol{x}_v(u, v) = (f'(v) \cos u, f'(v) \sin u, g'(v)).$$

The coefficients of the first fundamental form in this parametrization are

$$E(u, v) = f(v)^2, \quad F(u, v) = 0 \quad \text{and} \quad G(u, v) = |f'(v)|^2 + |g'(v)|^2 = \|\boldsymbol{\alpha}'(v)\|^2.$$

7.3 Arc lengths of a curve and angles between curves in a surface

The aim of the following remark is to calculate the arc length of a curve in a surface *using only the coefficients of the first fundamental form.*

Definition 7.13. Let $\boldsymbol{\alpha}\colon I \longrightarrow S$ be a curve on a regular surface S. Then the length of $\boldsymbol{\alpha}$, measured from a point $\boldsymbol{\alpha}(u_0)$ for some $u_0 \in I$, is

$$\ell(u) := \int_{u_0}^{u} \sqrt{\langle \boldsymbol{\alpha}'(s), \boldsymbol{\alpha}'(s)\rangle_{\boldsymbol{\alpha}(s)}}\,\mathrm{d}s.$$

Proposition 7.14 (evident)**.**

$$\ell(u) := \int_{u_0}^{u} [I_{\boldsymbol{\alpha}(s)}(\boldsymbol{\alpha}'(s))]^{1/2}\,\mathrm{d}s.$$

Remark 7.15. Let $\boldsymbol{\alpha}\colon I \longrightarrow S$ be a curve on a regular surface S and $\boldsymbol{x}\colon U \longrightarrow S$ a local parametrization such that $\boldsymbol{\alpha}(I) \subset \boldsymbol{x}(U)$. Denote by $\boldsymbol{\beta} = (u, v)$ the corresponding curve in the parameter domain (i.e., $\boldsymbol{\alpha}(s) = \boldsymbol{x}(\boldsymbol{\beta}(s)) = \boldsymbol{x}(u(s), v(s))$).

Let E, F, G be the coefficients of the first fundamental form w.r.t. the parametrization $\boldsymbol{x}$. Then the arc lengths of $\boldsymbol{\alpha}$ from $s_0 \in I$ to $s_1 \in I$ can be expressed in terms of E, F, G only as follows:

$$\ell(s_1) = \int_{s_0}^{s_1} [I_{\boldsymbol{\alpha}(t)}(\boldsymbol{\alpha}'(t))]^{1/2}\,\mathrm{d}t = \int_{s_0}^{s_1} \sqrt{u'(t)^2 E(\boldsymbol{\beta}(t)) + 2u'(t)v'(t)F(\boldsymbol{\beta}(t)) + v'(t)^2 G(\boldsymbol{\beta}(t))}\,\mathrm{d}t.$$

Example 7.16. The hyperbolic plane. We construct a surface by fixing the coefficients of the first fundamental form E, F, G only. Actually, this is the first example which cannot (in total) be realized as a surface in $\mathbb{R}^3$.

Let $U := \{\, (u, v) \in \mathbb{R}^2 \mid v > 0 \,\}$ be the upper halfplane and set

$$E(u, v) := \frac{1}{v^2}, \quad F(u, v) := 0 \quad \text{and} \quad G(u, v) := \frac{1}{v^2},$$

i.e., $F = 0$ and $E = G$.

Let us now assume that there is a surface S in an ambient space $\mathbb{R}^n$ and a parametrization $\boldsymbol{x}\colon U \longrightarrow S$ such that the corresponding coefficients of the fundamental form have the desired form.

Consider a curve $\boldsymbol{\alpha}\colon (0, \infty) \longrightarrow S$ given by $\boldsymbol{\alpha}(s) = \boldsymbol{x}(0, s)$. In the coordinates on U, the curve has the form $\boldsymbol{\beta}\colon (0, \infty) \longrightarrow U$, $\boldsymbol{\beta}(s) = (0, s)$. Then

$$\|\boldsymbol{\alpha}'(s)\|^2 = 0E(0, s) + 0 + 1G(0, s) = \frac{1}{s^2}$$

Therefore, the arc length of $\boldsymbol{\alpha}$ from $\boldsymbol{\alpha}(a)$ to $\boldsymbol{\alpha}(b)$ on S is

$$\int_a^b \|\boldsymbol{\alpha}'(s)\|\,\mathrm{d}s = \int_a^b \frac{1}{s}\,\mathrm{d}s = \log b - \log a = \log \frac{b}{a}.$$

The upper half-plane $U = \mathbb{R} \times (0, \infty)$ together with the first fundamental form above is called the *upper half-plane model of the hyperbolic plane.* The corresponding surface S, the *hyperbolic plane*, is sometimes denoted by $\mathbb{H}$.

Remark. Coordinate curves and angle. Let $\boldsymbol{x}\colon U \longrightarrow S$ be a parametrization of a regular surface $S \subset \mathbb{R}^n$, $(u_0, v_0) \in U$. Consider the curves

$$\boldsymbol{\alpha}_1(s) = \boldsymbol{x}(u_0 + s, v_0) \qquad \text{and} \qquad \boldsymbol{\alpha}_2(s) = \boldsymbol{x}(u_0, v_0 + s)$$

with s being small. These curves are called the *coordinate curves* of the parametrization $\boldsymbol{x}$. The angle formed by the two curves meeting in (u_0, v_0) can be calculated by

$$\cos\vartheta = \frac{\boldsymbol{\alpha}_1'(0)\cdot\boldsymbol{\alpha}_2'(0)}{\|\boldsymbol{\alpha}_1'(0)\|\|\boldsymbol{\alpha}_2'(0)\|}.$$

But $\boldsymbol{\alpha}_1'(0) = \boldsymbol{x}_u(u_0, v_0)$ and $\boldsymbol{\alpha}_2'(0) = \boldsymbol{x}_v(u_0, v_0)$, so that (omitting the argument (u_0, v_0))

$$\cos\vartheta = \frac{\boldsymbol{x}_u\cdot\boldsymbol{x}_v}{\|\boldsymbol{x}_u\|\|\boldsymbol{x}_v\|} = \frac{F}{\sqrt{EG}}.$$

7.4 Area of subsets of a surface

Definition 7.17. Let $R_0 \subset U$, $R = \boldsymbol{x}(R_0) \subset S$. The *area* of a region $R = \boldsymbol{x}(R_0)$ is defined as

$$\operatorname{area}(R) := \int_{R_0} \sqrt{EG - F^2}\,\mathrm{d}u\,\mathrm{d}v.$$

Example 7.18. Let S be a half of a cylinder parametrized by

$$\boldsymbol{x}(u,v) = (u, v, \sqrt{1-v^2}), \qquad (u,v) \in U = (-1,1)\times(-1,1)$$

Then $E \equiv 1$, $F \equiv 0$, $G = 1/(1-v^2)$, so

$$\operatorname{area}(S) = \int_U \sqrt{EG - F^2}\,\mathrm{d}u\,\mathrm{d}v = \int_{-1}^{1}\mathrm{d}u \int_{-1}^{1} \sqrt{1/(1-v^2)}\,\mathrm{d}v = 2\pi$$

The definition of area depends at first sight on the local parametrization $\boldsymbol{x}\colon U \longrightarrow S$. Actually, it does not:

Proposition 7.19. Assume that we have two local parametrizations $\boldsymbol{x}_1\colon U_1 \longrightarrow S$ and $\boldsymbol{x}_2\colon U_2 \longrightarrow S$ with $\boldsymbol{x}_1(U_1) = \boldsymbol{x}_2(U_2) =: W$. Denote by E_1, F_1, G_1 and E_2, F_2, G_2 the coefficients of the first fundamental form in the parametrisation $\boldsymbol{x}_1$ and $\boldsymbol{x}_2$, respectively.

Let $R \subset W$. Denote by $R_1 := \boldsymbol{x}_1^{-1}(R)$ and $R_2 := \boldsymbol{x}_2^{-1}(R)$ the corresponding regions in the respective parameter domains. Then

$$\int_{R_1} \sqrt{E_1G_1 - F_1^2}\,\mathrm{d}u_1\,\mathrm{d}v_1 = \int_{R_2} \sqrt{E_2G_2 - F_2^2}\,\mathrm{d}u_2\,\mathrm{d}v_2.$$

Example 7.20.

(a) **The sphere.** Let S be the sphere of radius $r > 0$ in $\mathbb{R}^3$,

$$\boldsymbol{x}(u,v) = (r\cos u\sin v, r\sin u\sin v, r\cos v)$$

(v measures *latitude*, u measures *longitude*, and (u,v) are called *spherical coordinates*). We have

$$E(u,v) = r^2\sin^2 v, \quad F(u,v) = 0 \quad \text{and} \quad G(u,v) = r^2,$$

so that $EG - F^2 = r^4\sin^2 v$.

Let us compute the area of a "slice" of the sphere enclosed by planes $z = z_0$ and $z = z_1$, where $-r \le z_1 < z_0 \le r$. This corresponds to the domain $\arccos z_0 \le v \le \arccos z_1, u \in (0, 2\pi)$. Therefore the area is

$$\int_0^{2\pi}\mathrm{d}u \int_{\arccos z_0}^{\arccos z_1} r^2\sin^2 v\,\mathrm{d}v = 2\pi r^2(z_0 - z_1).$$

(b) **Torus of revolution:** Consider the parametrization

$$\boldsymbol{x}\colon U := (0, 2\pi) \times (0, 2\pi) \longrightarrow S,$$
$$\boldsymbol{x}(u, v) := \big((R + r\cos v)\cos u, (R + r\cos v)\sin u, r\sin v\big)$$

for $0 < r < R$. This surface is a surface of revolution, obtained by rotating the curve $\boldsymbol{\alpha}$ given by

$$\boldsymbol{\alpha}(v) = \big((R + r\cos v), 0, r\sin v\big)$$

(which is a circle of radius r in the (x, z)-plane centered at the point $(R, 0, 0)$) around the z-axis. Then

$$\boldsymbol{x}_u(u, v) = \big(-(R + r\cos v)\sin u, (R + r\cos v)\cos u, 0\big),$$
$$\boldsymbol{x}_v(u, v) = \big(-r\sin v\cos u, -r\sin v\sin u, r\cos v\big)$$

and therefore

$$E(u, v) = (R + r\cos v)^2, \quad F(u, v) = 0 \quad \text{and} \quad G(u, v) = r^2.$$

In particular, $\sqrt{EG - F^2} = (R + r\cos v)r$, hence

$$\operatorname{area}(S) = \int_0^{2\pi}\int_0^{2\pi}(R + r\cos v)r\,\mathrm{d}u\,\mathrm{d}v = 4\pi^2 rR.$$

(c) **Hyperbolic plane:** Recall that we have the parameter domain $U := \mathbb{R} \times (0, \infty)$ together with the coefficients of the fundamental form

$$E(u, v) = G(u, v) = \frac{1}{v^2}, \quad F(u, v) = 0,$$

and $\sqrt{EG - F}(u, v) = 1/v^2$. Let $R_{a,b} := (0, b) \times (a, 2a)$, then the corresponding region in the hyperbolic plane $\mathbb{H}$ has area

$$\operatorname{area}(R) = \int_{R_{a,b}} \frac{1}{v^2}\,\mathrm{d}u\,\mathrm{d}v = \int_0^b \mathrm{d}u \int_a^{2a} \frac{1}{v^2}\,\mathrm{d}v = b/2a.$$

In particular, if $b = a$, we obtain $1/2$ which does not depend on a.

8 Smooth maps between surfaces

Recall that $f\colon U \longrightarrow \mathbb{R}^m$ is smooth at $p \in U$ if all partial derivatives of f at p exist and are continuous. We need $U \subset \mathbb{R}^n$ to be *open* to be able to define a partial derivative.

Let $S \subset \mathbb{R}^n$ be a regular surface and $f\colon S \longrightarrow \mathbb{R}^m$. Since S is not open in $\mathbb{R}^n$ $(n \geq 3)$, we need to define smoothness of f on S.

Definition 8.1. We say that $f\colon S \longrightarrow \mathbb{R}^m$ is smooth at p if

$$f \circ \boldsymbol{x}\colon U \longrightarrow \mathbb{R}^m$$

is smooth at q where $\boldsymbol{x}\colon U \longrightarrow S$ is a parametrization with $\boldsymbol{x}(q) = p$.

Remark 8.2. This definition does not depend on the parametrization $\boldsymbol{x}$. Indeed, if $\boldsymbol{y}\colon V \longrightarrow S$ is another parametrization (assume that $\boldsymbol{x}(U) = \boldsymbol{y}(V)$), then there exists a diffeomorphism $h\colon U \longrightarrow V$ such that $\boldsymbol{y} = \boldsymbol{x} \circ h$ (change of parameter). In particular, $f \circ \boldsymbol{y} = (f \circ \boldsymbol{x}) \circ h$ is also smooth.

8.1 The Gauss map

Let S be a regular surface in $\mathbb{R}^3$.

Definition 8.3. The *Gauss map*

$$\boldsymbol{N}\colon S \longrightarrow S^2$$

assigns, to each point $p \in S$, the unit normal to S at p, i.e., the unit vector orthogonal to $T_pS \subset \mathbb{R}^3$ (which is determined up to sign only!). Here, $S^2 := \{\, (x, y, z) \in \mathbb{R}^3 \mid x^2 + y^2 + z^2 = 1 \,\}$ is the unit sphere in $\mathbb{R}^3$.

In a local parametrization $\boldsymbol{x}\colon U \longrightarrow S$ of S, we have

$$\boldsymbol{N} \circ \boldsymbol{x}(u, v) := \frac{\boldsymbol{x}_u \times \boldsymbol{x}_v}{\|\boldsymbol{x}_u \times \boldsymbol{x}_v\|}(u, v),$$

and this map is always smooth.

Example 8.4.

(a) **Plane in $\mathbb{R}^3$:** $S = \{\, (x, y, z)\mathbb{R}^3 \mid ax + by + cz + d = 0 \,\}$. Then $\boldsymbol{N} = \frac{(a,b,c)}{\sqrt{a^2+b^2+c^2}} \equiv$ const.

(b) **Graph of a function:** $S = \{\, (u, v, g(u, v)) \mid (u, v) \in U \,\}$, $g\colon U \longrightarrow \mathbb{R}$ smooth, then $\boldsymbol{x}_u = (1, 0, g_u)$, $\boldsymbol{x}_v = (0, 1, g_v)$, then the Gauss map is given by $\boldsymbol{N}\colon S \longrightarrow S^2$

$$\boldsymbol{N} \circ \boldsymbol{x} = \frac{\boldsymbol{x}_u \times \boldsymbol{x}_v}{\|\boldsymbol{x}_u \times \boldsymbol{x}_v\|} = \frac{1}{\sqrt{1 + (g_u)^2 + (g_v)^2}}(-g_u, -g_v, 1).$$

As an example, take $g(u, v) = u^2 + v^2$, then

$$\boldsymbol{N}(\boldsymbol{x}(u, v)) = \frac{1}{\sqrt{1 + 4u^2 + 4v^2}}(-2u, -2v, 1)$$

Also, $S = f^{-1}(0)$ for $f(x, y, z) = x^2 + y^2 - z$, so $\nabla f = (2x, 2y, -1)$ is proportional to $\boldsymbol{N}$ as expected.

(c) **The catenoid:** $\boldsymbol{x}(u, v) = (\cosh v \cos u, \cosh v \sin u, v)$, then

$$\boldsymbol{x}_u(u, v) = (-\cosh v \sin u, \cosh v \cos u, 0) \quad \text{and} \quad \boldsymbol{x}_v(u, v) = (\sinh v \cos u, \sinh v \sin u, 1)$$

so that

$$(\boldsymbol{x}_u \times \boldsymbol{x}_v)(u, v) = (\cosh v \cos u, \cosh v \sin u, -\cosh v \sinh v),$$

and therefore

$$\boldsymbol{N}(\boldsymbol{x}(u, v)) = \frac{1}{\cosh v}(\cos u, \sin u, -\sinh v).$$

(d) **The sphere:** $\boldsymbol{N}\colon S^2 \longrightarrow S^2$ is given by $\boldsymbol{N}(p) = p$.

Remark. The Gauss map is well defined on $\boldsymbol{x}(U)$, but we may not be able to define it (continuously) on all S

Example 8.5. Möbius band

Definition 8.6. A surface in $\mathbb{R}^3$ is *non-orientable* if it is not possible to define the Gauss map globally.

Example 8.7. Further maps on surfaces. Let $S \subset \mathbb{R}^3$ be a surface.

(a) **Height function.** Fix $\boldsymbol{v} \in S^2$, and define a function $h\colon S \longrightarrow \mathbb{R}$ by $h(p) := p \cdot \boldsymbol{v}$. Then h is smooth. You can think of h measuring the height of S if you stand on the plane orthogonal to $\boldsymbol{v}$ fixed e.g. at the origin of $\mathbb{R}^3$.

(b) **Distance squared function.** Let $a \in \mathbb{R}^3$ and define $d^2\colon S \longrightarrow \mathbb{R}$ by $d^2(p) := \|p - a\|^2 = (p - a) \cdot (p - a)$, then d^2 is smooth. (d measures the distance of p from a *in the ambient space* $\mathbb{R}^3$).

8.2 The derivative of a smooth map between surfaces

Definition 8.8. Let S be a regular surface in $\mathbb{R}^\ell$, $p \in S$ and $f\colon S \longrightarrow \mathbb{R}^m$ a smooth map. The *derivative of f at p* is a linear map

$$d_p f\colon T_pS \longrightarrow \mathbb{R}^m$$

such that

$$d_pf(\boldsymbol{x}_u) = \partial_u(f \circ \boldsymbol{x})(q) \quad \text{and} \quad d_pf(\boldsymbol{x}_v) = \partial_v(f \circ \boldsymbol{x})(q)$$

for a local parametrization $\boldsymbol{x}\colon U \longrightarrow S$ of S with $\boldsymbol{x}(q) = p$, $q \in U \subset \mathbb{R}^2$. For short, we write

$$\boldsymbol{f}_u := d_pf(\boldsymbol{x}_u) \quad \text{and} \quad \boldsymbol{f}_v := d_pf(\boldsymbol{x}_v),$$

suppressing the local parametrisation $\boldsymbol{x}$ in the notation $\boldsymbol{f}_u$ and $\boldsymbol{f}_v$.

Remark 8.9.

(a) As $\{\boldsymbol{x}_u, \boldsymbol{x}_v\}$ is a basis of T_pS, and $\boldsymbol{w} \in T_p$ can be written as $\boldsymbol{w} = a\boldsymbol{x}_u + \boldsymbol{x}_v$, we have

$$d_pf(\boldsymbol{w}) = d_pf(a\boldsymbol{x}_u + b\boldsymbol{x}_v) = a d_pf(\boldsymbol{x}_u) + b d_pf(\boldsymbol{x}_v)$$

by the linearity of d_pf.

(b) d_pf does not depend on the choice of local parametrization $\boldsymbol{x}$. Indeed, if we take $\boldsymbol{w} \in T_pS$ and compute its image, then if $\boldsymbol{w} = \boldsymbol{\alpha}'(0)$ for $\boldsymbol{\alpha}\colon I \longrightarrow S$ a smooth curve, $\boldsymbol{\alpha}(0) = p$, we have $d_pf(\boldsymbol{w}) = (f \circ \alpha)'(0))$.

Example 8.10. (a) Let $S = \{\, (x, y, z) \in \mathbb{R}^3 \mid x^2 + y^2 = 1 \,\}$ be a cylinder in $\mathbb{R}^3$ and $f\colon S \longrightarrow \mathbb{R}$ be given by $f(p) = p \cdot p = \|p\|^2$. A local parametrization of S is given by

$$\boldsymbol{x}\colon U \longrightarrow S, \qquad \boldsymbol{x}(\vartheta, z) = (\cos\vartheta, \sin\vartheta, z), \qquad (\vartheta, r) \in U$$

Here, at least two parameter domains $U_1 = (0, 2\pi) \times \mathbb{R}$ and $U_2 = (-\pi, \pi) \times \mathbb{R}$ are needed in order to cover the entire cylinder. Then we have $(f \circ \boldsymbol{x})(\vartheta, z) = f(\cos\vartheta, \sin\vartheta, z)$ and

$$d_pf(\boldsymbol{x}_\vartheta) = \boldsymbol{f}_\vartheta = \frac{\partial}{\partial\vartheta}(f \circ \boldsymbol{x}) = 0 \quad \text{and} \quad d_pf(\boldsymbol{x}_z) = \boldsymbol{f}_z = \frac{\partial}{\partial z}(f \circ \boldsymbol{x}) = 2z.$$

(b) **(Gauss map of a catenoid)** Let S be parametrized by

$$\boldsymbol{x}(u, v) = (\cosh v \cos u, \cosh v \sin u, v),$$

then its Gauss map is given by

$$\boldsymbol{N}(\boldsymbol{x}(u, v)) = \frac{1}{\cosh v}\bigl(\cos u, \sin u, -\sinh v\bigr).$$

In particular, the derivative is

$$d_p\boldsymbol{N}(\boldsymbol{x}_u) = \boldsymbol{N}_u = \frac{1}{\cosh v}\bigl(-\sin u, \cos u, 0\bigr) \quad \text{and}$$
$$d_p\boldsymbol{N}(\boldsymbol{x}_v) = \boldsymbol{N}_v = \frac{1}{\cosh^2 v}\bigl(-\cos u \sinh v, -\sin u \sinh v, -1\bigr).$$

Proposition 8.11 (Chain Rule). Let $f\colon S_1 \longrightarrow S_2$ and $g\colon S_2 \longrightarrow S_3$ be smooth maps between the surfaces S_1, S_2 and S_3, then $g \circ f\colon S_1 \longrightarrow S_3$ is smooth and its derivative is given by

$$d_p(g \circ f) = d_{f(p)}g \circ d_pf\colon T_pS_1 \longrightarrow T_{g(f(p))}S_3$$

as linear maps, or pointwise,

$$d_p(g \circ f)(\boldsymbol{w}) = d_{f(p)}g(d_pf(\boldsymbol{w}))$$

for all $\boldsymbol{w} \in T_pS_1$ and $p \in S_1$.

8.3 Isometries and conformal maps

Let $S \subset \mathbb{R}^\ell$ be a regular surface. Recall that the *first fundamental form* (1^{st}FF) is given by

$$I_p\colon T_pS \longrightarrow \mathbb{R}, \qquad I_p(\boldsymbol{w}) = \langle \boldsymbol{w}, \boldsymbol{w} \rangle_{\mathbb{R}^\ell} = \|\boldsymbol{w}\|^2_{\mathbb{R}^\ell}.$$

Recall also that the 1^{st}FF is needed to calculate

- lengths of curves in S,
- angles between curves in S and
- the area of subsets of S.

Let now S and $\widetilde{S}$ be two surfaces with 1^{st}FFs I and $\widetilde{I}$, respectively, let $f\colon S \longrightarrow \widetilde{S}$ be a smooth map. If $d_pf\colon T_pS \longrightarrow T_{f(p)}\widetilde{S}$ "preserves" I_p and $\widetilde{I}_{f(p)}$, then these calculations should give the same result, i.e., S and $\widetilde{S}$ are bassically the same from a metric point of view (at least locally: see Example 8.13 (a) below)

Definition 8.12. Let $f\colon S \longrightarrow \widetilde{S}$ be a smooth map between two surfaces S and $\widetilde{S}$.

(a) The map f is called a *(local) isometry* if

$$\langle d_pf(\boldsymbol{w}_1), d_pf(\boldsymbol{w}_2) \rangle_{f(p)} = \langle \boldsymbol{w}_1, \boldsymbol{w}_2 \rangle_p$$

for all $\boldsymbol{w}_1, \boldsymbol{w}_2 \in T_pS$ and $p \in S$. The surfaces S and $\widetilde{S}$ are called *(locally) isometric* if there is a (local) isometry between them.

(b) The map f is called a *(global) isometry* if f is a local isometry and, additionally, $f\colon S \longrightarrow \widetilde{S}$ is *bijective.*

The surfaces S and $\widetilde{S}$ are called *(globally) isometric* if there is a (globally) isometry between them.

(c) The map f is called *conformal* if there is a smooth function

$$\lambda\colon S \longrightarrow (0, \infty)$$

such that

$$\langle d_pf(\boldsymbol{w}_1), d_pf(\boldsymbol{w}_2) \rangle_{f(p)} = \lambda(p) \langle \boldsymbol{w}_1, \boldsymbol{w}_2 \rangle_p$$

for all $\boldsymbol{w}_1, \boldsymbol{w}_2 \in T_pS$ and $p \in S$.

The surfaces S and $\widetilde{S}$ are called *conformally equivalent* if there is a conformal map between them.

Remark.

(a) Given a symmetric bilinear form $\langle , \rangle$, one can write $\langle \boldsymbol{w}_1, \boldsymbol{w}_2 \rangle = \frac{1}{2}(\|\boldsymbol{w}_1 + \boldsymbol{w}_2\|^2 - \|\boldsymbol{w}_1\|^2 - \|\boldsymbol{w}_2\|^2)$, which means that being a local isometry is equivalent to preserving 1^{st}FF, i.e. $\widetilde{I}_{f(p)}(d_pf(\boldsymbol{w})) = I_p(\boldsymbol{w})$, cf. Prop. 8.15.

(b) A conformal map with $\lambda \equiv 1$ is obviously a local isometry.

(c) A global isometry is obviously a local isometry, but not vice versa (see Example 8.13 (c) below).

(d) Conformal maps preserve angles. Indeed,

$$\vartheta = \angle(\boldsymbol{w}_1, \boldsymbol{w}_2) := \frac{\langle \boldsymbol{w}_1, \boldsymbol{w}_1\rangle_p}{\|\boldsymbol{w}_1\|_p \|\boldsymbol{w}_1\|_p} \quad \text{and}$$

$$\angle(d_pf(\boldsymbol{w}_1), d_pf(\boldsymbol{w}_2)) := \frac{\langle d_pf(\boldsymbol{w}_1), d_pf(\boldsymbol{w}_1)\rangle_p}{\|d_pf(\boldsymbol{w}_1)\|_p \|d_pf(\boldsymbol{w}_1\|_p)} = \frac{\lambda(p)\langle \boldsymbol{w}_1, \boldsymbol{w}_1\rangle_p}{\sqrt{\lambda(p)}\|\boldsymbol{w}_1\|_p \sqrt{\lambda(p)}\|\boldsymbol{w}_1\|_p} = \vartheta$$

since the factors involving $\lambda(p) > 0$ cancel each other.

(e) Local isometries preserve lengths of curves (but not distances between points). Global isometries preserve distances.

Example 8.13.

(a) Let $S = (0, 2\pi) \times \mathbb{R}$ and $\widetilde{S} = \{\,(x, y, z) \in \mathbb{R}^3 \,|\, x^2 + y^2 = 1\,\}$ (a cylinder). Define $f\colon S \longrightarrow \widetilde{S}$ by $f(\vartheta, z) = (\cos\vartheta, \sin\vartheta, z)$ for $p = (\vartheta, z) \in S$. We can think of S as being parametrized by itself (as a subset of the plane $\mathbb{R}^2$), and $T_pS = \mathbb{R}^2$.

One way to show that f is a local isometry is to ensure that it preserves 1^{st}FF (the identity matrix), which is an elementary computation of $\boldsymbol{f}_\vartheta$ and $\boldsymbol{f}_z$ and their dot products, cf. Prop. 8.15.

Alternatively, one can compute the differential of f explicitely. Write $\boldsymbol{w} = (a, b) \in T_pS$. We need $\boldsymbol{\alpha}\colon I \longrightarrow S$ with I being an open interval containing 0, $\boldsymbol{\alpha}(0) = p$ and $\boldsymbol{\alpha}'(0) = \boldsymbol{w}$. Take a line through $p \in S \subset \mathbb{R}^2$ in direction $\boldsymbol{w}$, i.e.

$$\boldsymbol{\alpha}(t) = p + t\boldsymbol{w} = (\vartheta + ta, z + tb).$$

Then

$$d_pf(\boldsymbol{w}) = d_pf(\boldsymbol{\alpha}'(0)) = (f \circ \boldsymbol{\alpha})'(0)$$

Here, we have

$$(f \circ \boldsymbol{\alpha})(t) = \big(\cos(\vartheta ta), \sin(\vartheta + ta), z + tb\big),$$

so that

$$(f \circ \boldsymbol{\alpha})'(0) = (-a\sin\vartheta, a\cos\vartheta, b) = d_pf(\boldsymbol{w}).$$

Now,

$$\langle d_pf(\boldsymbol{w}), d_pf(\boldsymbol{w})\rangle_{f(p)} = \langle(-a\sin\vartheta, a\cos\vartheta, b), (-a\sin\vartheta, a\cos\vartheta, b)\rangle = a^2 + b^2,$$

but we also have $\langle \boldsymbol{w}, \boldsymbol{w}\rangle_p = a^2 + b^2$, hence f is a local isometry.

(b) If we consider $f\colon S \longrightarrow \{\,(x, y, z) \,|\, x^2 + y^2 = 1, (x, y) \neq (1, 0)\,\}$, then f is bijective (check this!) and f is indeed a *global* isometry.

(c) If we consider $f\colon \mathbb{R} \times \mathbb{R} \longrightarrow \widetilde{S}$ (with the same definition of $f(\vartheta, z)$ as before, but now $\vartheta \in \mathbb{R}$), then f is still a local isometry (the calculation remains the same as above), but not a *global* isometry: f is no longer injective and hence not bijective.

Example 8.14 (Conformal bijections of $\mathbb{R}^2$). As one can recall from Complex Analysis, conformal maps are holomorphic (or anti-holomorphic) and vise versa. Thus conformal bijections of the plane are holomorphic one-to-one maps. They must have a single pole at infinity, so they are polynomial of degree one (possibly with conjugation), i.e. $f(z) = az + b$ or $f(z) = a\bar{z} + b$, $a, b \in \mathbb{C}$, $a \neq 0$. The conformal factor is $\lambda(z) = |a|^2$.

Proposition 8.15. Let S, $\widetilde{S}$ be two surfaces and $\boldsymbol{x}\colon U \longrightarrow S$ be a local parametrization of S.

A map $f\colon S \longrightarrow \widetilde{S}$ is a local isometry on $\boldsymbol{x}(U)$ if and only if

$$\langle \boldsymbol{f}_u, \boldsymbol{f}_u\rangle = E, \quad \langle \boldsymbol{f}_u, \boldsymbol{f}_v\rangle = F \quad \text{and} \quad \langle \boldsymbol{f}_v, \boldsymbol{f}_v\rangle = G, \tag{8.-2}$$

where E, F, G are the coefficients of the 1^{st}FF w.r.t. $\boldsymbol{x}$. Here $\boldsymbol{f}_u = \partial_u(f \circ \boldsymbol{x})$ and $\boldsymbol{f}_v = \partial_v(f \circ \boldsymbol{x})$ and $(u, v) \in U$ are the parameter coordinates).

Remark.

(a) If we denote by $\widetilde{E}$, $\widetilde{F}$ and $\widetilde{G}$ the coefficients of the 1^{st}FF of $\widetilde{S}$ w.r.t. the parametrization $\widetilde{\boldsymbol{x}} = f \circ \boldsymbol{x}\colon U \longrightarrow \widetilde{S}$, then we can rephrase this as

$$\widetilde{E} = E, \quad \widetilde{F} = F \quad \text{and} \quad \widetilde{G} = G.$$

(b) A similar result holds for conformal maps: f is conformal on $\boldsymbol{x}(U)$ iff there exists a smooth map $\mu\colon U \longrightarrow (0, \infty)$ such that

$$\langle \boldsymbol{f}_u, \boldsymbol{f}_u\rangle = \mu E, \quad \langle \boldsymbol{f}_u, \boldsymbol{f}_v\rangle = \mu F \quad \text{and} \quad \langle \boldsymbol{f}_v, \boldsymbol{f}_v\rangle = \mu G,$$

Example 8.16. (a) Spheres of distinct radii are conformally equivalent (but not isometric, will see this later).

(b) **Gauss map of the catenoid is conformal.** We have seen in Example 8.10 (b) and previous examples that for the parametrization $\boldsymbol{x}$ given by

$$\boldsymbol{x}(u, v) = (\cosh v \cos u, \cosh v \sin u, v),$$

the coefficients of the 1^{st}FF are

$$E = G = \cosh^2 v \quad \text{and} \quad F = 0.$$

Moreover, the derivatives of the Gauss map are

$$\boldsymbol{N}_u = \frac{1}{\cosh v}\begin{pmatrix} -\sin u \\ \cos u \\ 0 \end{pmatrix} \quad \text{and} \quad \boldsymbol{N}_v = \frac{1}{\cosh^2 v}\begin{pmatrix} -\cos u \sinh v \\ -\sin u \sinh v \\ -1 \end{pmatrix}.$$

Now,

$$\begin{aligned} \langle \boldsymbol{N}_u, \boldsymbol{N}_u\rangle &= \frac{1}{\cosh^2 v} = \frac{1}{\cosh^4 v}E, \quad \langle \boldsymbol{N}_u, \boldsymbol{N}_v\rangle = 0 = F \quad \text{and} \\ \langle \boldsymbol{N}_v, \boldsymbol{N}_v\rangle &= \frac{\sinh^2 v + 1}{\cosh^4 v} = \frac{1}{\cosh^2 v} = \frac{1}{\cosh^4 v}G, \end{aligned}$$

so $\boldsymbol{N}$ is a conformal map with conformal factor (in local parametrization) μ given by $\mu(u, v) = 1/\cosh^4(v)$.

9 Geometry of the Gauss map

9.1 The Weingarten map

Lemma 9.1. Let S be a surface in $\mathbb{R}^3$ and $\boldsymbol{N}\colon S \longrightarrow S^2$ be its Gauss map. Then $d_p\boldsymbol{N}(\boldsymbol{w})$ is orthogonal to $\boldsymbol{N}(p)$ for every $\boldsymbol{w} \in T_pS$. In particular, we can identify $T_{\boldsymbol{N}(p)}S^2$ and T_pS, and consider $d_p\boldsymbol{N}$ as a map

$$d_p\boldsymbol{N}\colon T_pS \longrightarrow T_pS.$$

Moreover, $d_p\boldsymbol{N}$ is *symmetric*, i.e.,

$$\langle d_p\boldsymbol{N}(\boldsymbol{w}_1), \boldsymbol{w}_2\rangle = \langle \boldsymbol{w}_1, d_p\boldsymbol{N}(\boldsymbol{w}_2)\rangle$$

for all $\boldsymbol{w}_1, \boldsymbol{w}_2 \in T_pS$.

Definition 9.2. (a) The map $-d_p\boldsymbol{N}\colon T_pS \longrightarrow T_pS$ is called the *Weingarten map* of the surface $S \subset \mathbb{R}^3$ at $p \in S$.

(b) The quadratic form $I\!I_p\colon T_pS \longrightarrow \mathbb{R}$, $I\!I_p(\boldsymbol{w}) = \langle -d_p\boldsymbol{N}(\boldsymbol{w}), \boldsymbol{w}\rangle$, is called the *second fundamental form of S at p.*

Remark 9.3. Since $-d_p\boldsymbol{N}$ is symmetric, the Weingarten map is diagonalizable in an orthogonal basis of T_pS.

Since $-d_p\boldsymbol{N}$ is now a linear operator on the tangent space T_pS, we can calculate its characteristic polynomial, trace, determinant and eigenvalues (these do not depend on a basis).

Definition 9.4. Let S be a regular surface in $\mathbb{R}^3$ with Gauss map $\boldsymbol{N}\colon S \longrightarrow S^2$ and Weingarten map $-d_p\boldsymbol{N}\colon T_pS \longrightarrow T_pS$ at $p \in S$.

(a) $K(p) = \det(-d_p\boldsymbol{N})$ is called the *Gauss curvature of S at p.*

(b) $H(p) = \frac{1}{2}\operatorname{tr}(-d_p\boldsymbol{N})$ is called the *mean curvature of S at p.*

(c) The eigenvalues $\kappa_1(p)$, $\kappa_2(p)$ of $-d_p\boldsymbol{N}$ are called *principal curvatures* of S at p.

(d) The eigenvectors $\boldsymbol{e}_1(p)$, $\boldsymbol{e}_2(p)$ of $-d_p\boldsymbol{N}$ are called *principal directions* of S at p (i.e., $-d_p\boldsymbol{N}(\boldsymbol{e}_i(p)) = \kappa_i(p)\boldsymbol{e}_i(p)$).

Remark 9.5. Obviously, we have

$$K(p) = \kappa_1(p)\kappa_2(p), \quad H(p) = \frac{1}{2}\big(\kappa_1(p) + \kappa_2(p)\big).$$

Example 9.6 (Sphere)**.** Let $S = S^2(r)$ for some $r > 0$ be a sphere. The normal vector at $\boldsymbol{p} \in S$ is given by

$$\boldsymbol{N}(\boldsymbol{p}) = \frac{1}{r}\,\boldsymbol{p}.$$

Thus, the Weingarten map is a scalar operator

$$-d_{\boldsymbol{p}}\boldsymbol{N}(\boldsymbol{w}) = -\frac{1}{r}\,\boldsymbol{w}.$$

In particular, the second fundamental form is

$$I\!I_p(\boldsymbol{w}) = \langle -d_p\boldsymbol{N}(\boldsymbol{w}), \boldsymbol{w}\rangle = -\frac{1}{r}\|\boldsymbol{w}\|^2.$$

Moreover, the eigenvalues are $\kappa_1(p) = \kappa_2(p) = -1/r$, the Gauss curvature is $K(p) = 1/r^2$ and the mean curvature is $H(p) = -1/r$.

Definition 9.7. Let S be a regular surface in $\mathbb{R}^3$ with Gauss map $\boldsymbol{N}\colon S \longrightarrow S^2$, and let $\boldsymbol{x}\colon U \longrightarrow S$ be a local parametrization. We call

$$L = \boldsymbol{x}_{uu} \cdot \boldsymbol{N}, \qquad M = \boldsymbol{x}_{uv} \cdot \boldsymbol{N} \quad \text{and} \quad N = \boldsymbol{x}_{vv} \cdot \boldsymbol{N}$$

the *coefficients of the second fundamental form.*

Proposition 9.8. L, M, N are indeed the coefficients of $I\!I_p$ in the basis $\{\boldsymbol{x}_u, \boldsymbol{x}_v\}$, i.e.

$$I\!I_p(a\boldsymbol{x}_u + b\boldsymbol{x}_v) = a^2 L + 2abM + b^2 N$$

Computing the matrix of the Weingarten map in the basis $\{\boldsymbol{x}_u, \boldsymbol{x}_v\}$ gives a matrix

$$-d_p\boldsymbol{N} = \frac{1}{EG - F^2}\begin{pmatrix} GL - FM & GM - FN \\ -FL + EM & -FM + EN \end{pmatrix},$$

which results in the following.

Proposition 9.9.

$$K = \frac{LN - M^2}{EG - F^2}, \qquad H = \frac{1}{2}\frac{EN - 2FM + GL}{EG - F^2}.$$

Example 9.10. Hyperbolic paraboloid.

Let $S := \{\, (x, y, z) \,|\, x^2 - y^2 + z = 0 \,\}$. It may be parametrized as a graph of a function $z = f(x, y) = y^2 - x^2$, i.e., $\boldsymbol{x}(u, v) = (u, v, v^2 - u^2)$ for $(u, v) \in U = \mathbb{R}^2$. Then

$$\begin{aligned} \boldsymbol{x}_u &= (1, 0, -2u), \qquad \boldsymbol{x}_v = (0, 1, 2v), \\ \boldsymbol{x}_{uu} &= (0, 0, -2), \quad \boldsymbol{x}_{uv} = (0, 0, 0), \quad \boldsymbol{x}_{vv} = (0, 0, 2). \end{aligned}$$

We also need the normal and calculate

$$\boldsymbol{x}_u \times \boldsymbol{x}_v = (2u, -2v, 1),$$

which has norm $D = (4u^2 + 4v^2 + 1)^{1/2}$, hence

$$\boldsymbol{N} \circ \boldsymbol{x} = \frac{1}{D}(2u, -2v, 1).$$

The coefficients of the 1^{st}FF and 2^{nd}FF are

$$\begin{aligned} E &= \boldsymbol{x}_u \cdot \boldsymbol{x}_u = 1 + 4u^2, \quad F = \boldsymbol{x}_u \cdot \boldsymbol{x}_v = -4uv, \quad G = \boldsymbol{x}_v \cdot \boldsymbol{x}_v = 1 + 4v^2 \\ L &= \boldsymbol{x}_{uu} \cdot \boldsymbol{N} = \frac{-2}{D}, \quad M = \boldsymbol{x}_{uv} \cdot \boldsymbol{N} = 0, \quad N = \boldsymbol{x}_{vv} \cdot \boldsymbol{N} = \frac{2}{D}. \end{aligned}$$

Now,

$$EG - F^2 = (1 + 4u^2)(1 + 4v^2) - 16u^2v^2 = 1 + 4u^2 + 4v^2 = D^2 \quad \text{and} \quad LN - M^2 = \frac{-4}{D^2},$$

so that the Gauss curvature is

$$K = \frac{LN - M^2}{EG - F^2} = \frac{-4}{D^4} < 0$$

and the mean curvature is

$$H = \frac{EN + GL}{2(EG - F^2)} = \frac{(1 + 4u^2) - (1 + 4v^2)}{D^3} = \frac{4(u^2 - v^2)}{D^3}.$$

Let us calculate the principal curvatures at $\boldsymbol{x}(0,0) = (0,0,0)$ (i.e., $(u,v) = (0,0)$). Here, $K = -4$ and $H = 0$, hence we look for the roots κ of

$$\kappa^2 - 2H\kappa + K = 0, \quad \text{or,} \quad \kappa^2 - 4 = 0,$$

i.e., $\kappa_1 = 2$ and $\kappa_2 = -2$.

Definition 9.11. A parametrization $\boldsymbol{x}$ with $F = 0$ is called *orthogonal*, a parametrization $\boldsymbol{x}$ with $F = 0$ and $M = 0$ is called *principal*.

Proposition 9.12. Assume that the parametrization $\boldsymbol{x}$ of a surface is principal (i.e., $F = 0$ and $M = 0$), then $\boldsymbol{x}_u$ and $\boldsymbol{x}_v$ are the principal directions. Moreover, the principal curvatures are

$$\kappa_1 = \frac{L}{E} \qquad \text{and} \qquad \kappa_2 = \frac{N}{G}.$$

Hence, the Gauss and mean curvatures are

$$K = \kappa_1\kappa_2 = \frac{LN}{EG} \qquad \text{and} \qquad H = \frac{1}{2}(\kappa_1 + \kappa_2) = \frac{GL + EN}{2EG}.$$

Example 9.13. Surface of revolution. Let S be obtained by rotating the curve given by $\boldsymbol{\alpha}(v) = (f(v), 0, g(v))$, $v \in I$ (some open interval) around the z-axis. Let us assume that $f(v) > 0$. A local parametrization is then given by

$$\boldsymbol{x}(u,v) = \begin{pmatrix} f(v)\cos u \\ f(v)\sin u \\ g(v) \end{pmatrix}$$

for $(u,v) \in U_1 = (0, 2\pi) \times I$ (and $(u,v) \in U_2 = (-\pi, \pi) \times I$ to cover the surface entirely). The derivatives are

$$\boldsymbol{x}_u = \begin{pmatrix} -f(v)\sin u \\ f(v)\cos u \\ 0 \end{pmatrix} \quad \text{and} \quad \boldsymbol{x}_v = \begin{pmatrix} f'(v)\cos u \\ f'(v)\sin u \\ g'(v) \end{pmatrix}.$$

For the coefficients of the second fundamental form, we also need the *second derivatives* of $\boldsymbol{x}$:

$$\boldsymbol{x}_{uu} = \begin{pmatrix} -f(v)\cos u \\ -f(v)\sin u \\ 0 \end{pmatrix}, \quad \boldsymbol{x}_{uv} = \boldsymbol{x}_{vu} = \begin{pmatrix} -f'(v)\sin u \\ f'(v)\cos u \\ 0 \end{pmatrix} \quad \text{and} \quad \boldsymbol{x}_{vv} = \begin{pmatrix} f''(v)\cos u \\ f''(v)\sin u \\ g''(v) \end{pmatrix}.$$

The normal vector at $p = \boldsymbol{x}(u,v)$ is

$$\boldsymbol{N}(p) = \Big(\frac{1}{\|\boldsymbol{x}_u \times \boldsymbol{x}_v\|}\boldsymbol{x}_u \times \boldsymbol{x}_v\Big)(u,v) = \frac{1}{\boldsymbol{\alpha}'(v)} \begin{pmatrix} g'(v)\cos u \\ g'(v)\sin u \\ -f'(v) \end{pmatrix},$$

where $\|\boldsymbol{\alpha}'(v)\| = (f'(v)^2 + g'(v)^2)^{1/2}$. Now, the coefficients of the second fundamental form are

$$L = \boldsymbol{x}_{uu} \cdot \boldsymbol{N} = \frac{-fg'}{\|\boldsymbol{\alpha}'\|}, \qquad M = \boldsymbol{x}_{uv} \cdot \boldsymbol{N} = 0 \quad \text{and}$$
$$N = \boldsymbol{x}_{vv} \cdot \boldsymbol{N} = \frac{f''g' - f'g''}{\|\boldsymbol{\alpha}'\|}.$$

The coefficients of the 1^{st}FF

$$E = \langle \boldsymbol{x}_u, \boldsymbol{x}_u \rangle = f^2, \quad F = \langle \boldsymbol{x}_u, \boldsymbol{x}_v \rangle = 0 \quad \text{and} \quad G = \langle \boldsymbol{x}_u, \boldsymbol{x}_v \rangle = \|\boldsymbol{\alpha}'\|^2.$$

Now we can calculate all the curvatures. The principal curvatures are

$$\kappa_1 = \frac{L}{E} = \frac{-fg'}{f^2\|\boldsymbol{\alpha}'\|} = \frac{-g'}{f\|\boldsymbol{\alpha}'\|} \qquad \text{and} \qquad \kappa_2 = \frac{N}{G} = \frac{f''g' - f'g''}{\|\boldsymbol{\alpha}'\|^3}.$$

Hence, the Gauss and mean curvatures are

$$K = \kappa_1\kappa_2 = \frac{LN}{EG} = \frac{-g'(f''g' - f'g'')}{f\|\boldsymbol{\alpha}'\|^4} \quad \text{and}$$
$$H = \frac{1}{2}(\kappa_1 + \kappa_2) = \frac{-g'}{2f} + \frac{f''g' - f'g''}{2\|\boldsymbol{\alpha}'\|^3}.$$

Example 9.14. Torus of revolution. Apply the above to the case $f(v) = R + r\cos(v/r)$ and $g(v) = r\sin(v/r)$, $0 < r < R$. Calculate the principal, Gauss curvature and mean curvatures.

We just calculate

$$f'(v) = -\sin(v/r), \qquad g'(v) = \cos(v/r),$$
$$f''(v) = -\frac{1}{r}\cos(v/r), \qquad g''(v) = -\frac{1}{r}\sin(v/r).$$

so that

$$\kappa_1 = \frac{-g'}{f} = \frac{\cos}{R + r\cos} \qquad \text{and} \qquad \kappa_2 = \frac{f''g' - f'g''}{f} = -\frac{1}{r}(\cos^2 + \sin^2) = -\frac{1}{r}$$

since $(f')^2 + (g')^2 = 1$ (the arguments of cos and sin in this formula are v/r). In particular, one principal curvature is constant (it is the one coming from going around the torus along the small circle, i.e., in direction $\boldsymbol{x}_u$). Moreover,

$$K = \kappa_1\kappa_2 = \frac{\cos}{r(R + r\cos)} \qquad \text{and} \qquad H = \frac{\cos}{2(R + r\cos)} - \frac{1}{2r} = \frac{-R}{2r(R + r\cos)}.$$

Note that the mean curvature never vanishes.

Definition 9.15.

(a) Let S be a surface and $K(p)$ its Gauss curvature at $p \in S$. We say that p is

$$\begin{cases} \textit{elliptic} & \text{if} \quad K(p) > 0 \\ \textit{hyperbolic} & \text{if} \quad K(p) < 0 \\ \textit{flat} & \text{if} \quad K(p) = 0 \end{cases}$$

The subset $\begin{cases} \{\, p \in S \mid K(p) > 0 \,\} \\ \{\, p \in S \mid K(p) < 0 \,\} \\ \{\, p \in S \mid K(p) = 0 \,\} \end{cases}$ is called $\begin{cases} \textit{elliptic} \\ \textit{hyperbolic} \\ \textit{flat} \end{cases}$ region of S

(b) Denote by $\kappa_1(p)$ and $\kappa_2(p)$ the principal curvatures at $p \in S$.

- We say that p is *planar* if $\kappa_1(p) = 0$ and $\kappa_2(p) = 0$;

- we say that p is *umbilic* if $\kappa_1(p) = \kappa_2(p)$.

Example 9.16. (a) (Sphere) On a sphere $S^2(r)$, all points are elliptic and umbilic since both principal curvatures are $\kappa_1(p) = \kappa_2(p) = -1/r$. The converse is also true (see Theorem 9.19).

(b) (Plane) It is not hard to see that if S is a plane (or an open subset of it) then all points of S are planar. The converse is also true (see Theorem 9.19).

(c) (Hyperbolic paraboloid, Example 9.10) All points are hyperbolic (since $K(p) < 0$ for all $p \in S$), and in particular, there are no umbilic points or flat points.

(d) (Torus of revolution, Example 9.14) We have $K = 0$ iff $\cos(v/r) = 0$ i.e., if $v/r = \pi/2$ or $v/r = 3\pi/2$. This is the circle on top and bottom of the torus; this is the *flat region*. The *elliptic region* is given by points with $K > 0$, i.e., $-\pi/2 < v/r < -\pi/2$. The *hyperbolic region* is given by points with $K < 0$, i.e., $\pi/2 < v/r < 3\pi/2$.

There are no umbilic points on the torus of revolution: $|\kappa_1| < 1/r$, but $\kappa_2 = -1/r$, so the two principal curvatures cannot be the same. There are no planar points either $\kappa_2 = -1/r \neq 0$ everywhere).

9.2 Some global theorems about curvature

Theorem 9.17. Every compact surface in $\mathbb{R}^3$ has at least one elliptic point.

Remark 9.18. The theorem is obviously false if either boundedness or closedness is dropped.

Theorem 9.19. Let S be a surface in $\mathbb{R}^3$.

(a) If all points of S are umbilic and $K \neq 0$ in at least one point of S then S is a part of a sphere.

(b) If all points of S are planar then S is part of a plane.

Theorem 9.20 (Conjecture of Carathéodory)**.** Every compact surface in $\mathbb{R}^3$ (convex, homeomorphic to a sphere) has at least two umbilic points.

This theorem has recently (2008) been proved (with additional smoothness assumptions) by Brendan Guilfoyle and Wilhelm Klingenberg (Durham).

Definition 9.21. A surface S is *minimal* if the mean curvature H vanishes identically on S.

10 The Theorema Egregium of Gauss

"Theorema Egregium" means "Remarkable Theorem".

Theorem 10.1 (Theorema Egregium)**.** The Gauss curvature of a surface in $\mathbb{R}^3$ depends on E, F, G and their derivatives only (in a local parametrization).

In other words: the Gauss curvature is *intrinsic*.

Corollary 10.2. A local isometry preserves the Gauss curvature.

The converse is false: a map preserving the Gauss curvature is not necessarily a (local) isometry, see Remark 10.11.

Remark 10.3. Theorem 10.1 does *not* hold for the mean curvature: e.g. $H = 0$ (plane) but $H = 1/(2r)$ (cylinder), although the plane and the cylinder are locally isometric.

Definition 10.4 (Christoffel symbols)**.** Let $\boldsymbol{x}\colon U \longrightarrow S$ be a local parametrization of a surface S in $\mathbb{R}^3$. The Christoffel symbols Γ_{ij}^k $(i,j,k \in \{1,2\})$ are functions $\Gamma_{ij}^k\colon U \longrightarrow \mathbb{R}$ defined by

$$\begin{aligned}
\boldsymbol{x}_{uu} &= \Gamma_{11}^1\boldsymbol{x}_u + \Gamma_{11}^2\boldsymbol{x}_v + L\boldsymbol{N}\\
\boldsymbol{x}_{uv} &= \Gamma_{12}^1\boldsymbol{x}_u + \Gamma_{12}^2\boldsymbol{x}_v + M\boldsymbol{N}\\
\boldsymbol{x}_{vu} &= \Gamma_{21}^1\boldsymbol{x}_u + \Gamma_{21}^2\boldsymbol{x}_v + M\boldsymbol{N}\\
\boldsymbol{x}_{vv} &= \Gamma_{22}^1\boldsymbol{x}_u + \Gamma_{22}^2\boldsymbol{x}_v + N\boldsymbol{N}
\end{aligned}$$

In particular, $\Gamma_{ij}^k = \Gamma_{ji}^k$.

Lemma 10.5.

(a) We have the identities

$$\begin{aligned}
\boldsymbol{x}_{uu}\cdot\boldsymbol{x}_u &= \frac{1}{2}E_u & \boldsymbol{x}_{vv}\cdot\boldsymbol{x}_v &= \frac{1}{2}G_v\\
\boldsymbol{x}_{uv}\cdot\boldsymbol{x}_u &= \frac{1}{2}E_v & \boldsymbol{x}_{uv}\cdot\boldsymbol{x}_v &= \frac{1}{2}G_u\\
\boldsymbol{x}_{vv}\cdot\boldsymbol{x}_u &= F_v - \frac{1}{2}G_u & \boldsymbol{x}_{uu}\cdot\boldsymbol{x}_v &= F_u - \frac{1}{2}E_v
\end{aligned}$$

for the coefficients E, F and G of the first fundamental form with respect to a parametrization $\boldsymbol{x}$.

(b) The Christoffel symbols are uniquely determined by E, F, G and their first derivatives.

Corollary 10.6. Gauss' Theorema Egregium allows us to define the Gauss curvature for *any* surface S just using the *first fundamental form.*

Example 10.7 (Gauss curvature of the hyperbolic plane)**.** Recall that we define the hyperbolic plane as a surface $\mathbb{H}$ parametrized by $x\colon U \longrightarrow H$ with

$$U = \mathbb{R}\times(0,\infty), \qquad E(u,v) = G(u,v) = \frac{1}{v^2}, \quad F(u,v) = 0.$$

Step 1 — Christoffel symbols: We first calculate the Christoffel symbols in the case that $F = 0$ (you can read off Γ_{ij}^k directly):

$$\begin{cases} E\Gamma_{11}^1 &= \frac{1}{2}E_u\\ G\Gamma_{11}^2 &= -\frac{1}{2}E_v\end{cases} \qquad \begin{cases} E\Gamma_{12}^1 &= \frac{1}{2}E_v\\ G\Gamma_{12}^2 &= \frac{1}{2}G_u\end{cases} \qquad \begin{cases} E\Gamma_{22}^1 &= -\frac{1}{2}G_u\\ G\Gamma_{22}^2 &= \frac{1}{2}G_v\end{cases}$$

or in our case (E and G are functions of v only).

$$\begin{cases} \frac{1}{v^2}\Gamma_{11}^1 &= 0\\ \frac{1}{v^2}\Gamma_{11}^2 &= \frac{1}{v^3}\end{cases} \qquad \begin{cases} \frac{1}{v^2}\Gamma_{12}^1 &= -\frac{1}{v^3}\\ \frac{1}{v^2}\Gamma_{12}^2 &= 0\end{cases} \qquad \begin{cases} \frac{1}{v^2}\Gamma_{22}^1 &= 0\\ \frac{1}{v^2}\Gamma_{22}^2 &= -\frac{1}{v^3}\end{cases}$$

or

$$\begin{cases} \Gamma_{11}^1 &= 0\\ \Gamma_{11}^2 &= \frac{1}{v}\end{cases} \qquad \begin{cases} \Gamma_{12}^1 &= -\frac{1}{v}\\ \Gamma_{12}^2 &= 0\end{cases} \qquad \begin{cases} \Gamma_{22}^1 &= 0\\ \Gamma_{22}^2 &= -\frac{1}{v}.\end{cases}$$

Therefore,

$$\begin{aligned}
\boldsymbol{x}_{uu} &= \Gamma_{11}^1 \boldsymbol{x}_u + \Gamma_{11}^2 \boldsymbol{x}_v + L\boldsymbol{N} = \frac{1}{v}\boldsymbol{x}_v + L\boldsymbol{N}\\
\boldsymbol{x}_{uv} &= \Gamma_{12}^1 \boldsymbol{x}_u + \Gamma_{12}^2 \boldsymbol{x}_v + M\boldsymbol{N} = -\frac{1}{v}\boldsymbol{x}_u + M\boldsymbol{N}\\
\boldsymbol{x}_{vv} &= \Gamma_{22}^1 \boldsymbol{x}_u + \Gamma_{22}^2 \boldsymbol{x}_v + N\boldsymbol{N} = -\frac{1}{v}\boldsymbol{x}_v + N\boldsymbol{N}
\end{aligned}$$

Step 2 — Calculate $LN - M^2$:

$$\begin{aligned}
LN - M^2 &= L\boldsymbol{N}\cdot N\boldsymbol{N} - M\boldsymbol{N}\cdot M\boldsymbol{N}\\
&= (\boldsymbol{x}_{uu} - \frac{1}{v}\boldsymbol{x}_v)\cdot(\boldsymbol{x}_{vv} + \frac{1}{v}\boldsymbol{x}_v) - (\boldsymbol{x}_{uv} + \frac{1}{v}\boldsymbol{x}_u)\cdot(\boldsymbol{x}_{uv} + \frac{1}{v}\boldsymbol{x}_u)\\
&= \boldsymbol{x}_{uu}\cdot\boldsymbol{x}_{vv} - \boldsymbol{x}_{uv}\cdot\boldsymbol{x}_{uv} \qquad - \frac{1}{v}\underbrace{\boldsymbol{x}_{vv}\cdot\boldsymbol{x}_v}_{=G_v/2=-1/v^3} + \frac{1}{v}\underbrace{\boldsymbol{x}_{uu}\cdot\boldsymbol{x}_v}_{=F_u-E_v/2=1/v^3} - \frac{1}{v^2}\underbrace{\boldsymbol{x}_v\cdot\boldsymbol{x}_v}_{=G=1/v^2}\\
&\qquad - 2\frac{1}{v}\underbrace{\boldsymbol{x}_{uv}\cdot\boldsymbol{x}_u}_{=E_v/2=-1/v^3} - \frac{1}{v^2}\underbrace{\boldsymbol{x}_u\cdot\boldsymbol{x}_u}_{=E=1/v^2}\\
&= \boldsymbol{x}_{uu}\cdot\boldsymbol{x}_{vv} - \boldsymbol{x}_{uv}\cdot\boldsymbol{x}_{uv} \qquad + \frac{2}{v^4}.
\end{aligned}$$

We now have

$$\begin{aligned}
\boldsymbol{x}_{uu}\cdot\boldsymbol{x}_{vv} - \boldsymbol{x}_{uv}\cdot\boldsymbol{x}_{uv} &= (\boldsymbol{x}_u\cdot\boldsymbol{x}_{vv})_u - (\boldsymbol{x}_u\cdot\boldsymbol{x}_{uv})_v\\
&= (F_v - \frac{1}{2}G_u)_u - \frac{1}{2}E_{vv} = -\frac{\partial^2}{\partial v^2}\frac{1}{2v^2} = -\frac{3}{v^4}.
\end{aligned}$$

Step 3 — Calculate K: Since $EG - F^2 = 1/v^4$, we have finally

$$K = \frac{LN - M^2}{EG - F^2} = \frac{-3/v^4 + 2/v^4}{1/v^4} = -1.$$

As a result, we have: the hyperbolic plane has constant curvature -1.

Remark 10.8.

(a) In Example 10.7 (or more generally, in all examples where we calculate the Gauss curvature from E, F and G only) we used the fact that $S \subset \mathbb{R}^3$ (at least locally), because we used the formulae for $\boldsymbol{x}_{uu}$ etc. involving the normal vector $\boldsymbol{N}$. This is for convenience only, to remember the procedure. More precisely, we should use the formula

$$K = \Big(\frac{LN - M^2}{EG - F^2} =\Big)\frac{E_{vv}/2 + F_{uv} - E_{vv}/2 + \text{terms in } E\text{,}F\text{,}G \text{ and derivatives}}{EG - F^2}$$

as *the definition* of K for a general surface as we did in Theorem 10.1.

(b) Recall that for plane curves the signed curvature defined a curve up to an isometry of the plane. What about a similar result for surfaces? Does the Gauss curvature define a surface uniquely (or up to what data the surface is unique)?

The answer to the uniqueness is *negative*, as Remark 10.11 shows: there exist surfaces S, $\widetilde{S}$ and a diffeomorphism $f\colon S \longrightarrow \widetilde{S}$ (f is bijective, smooth and f^{-1} is also smooth) which is *not* an isometry, but for which the Gauss curvature is preserved (i.e., $K(p) = \widetilde{K}(f(p))$, if K resp. $\widetilde{K}$ is the Gauss curvature of S resp. $\widetilde{S}$).

Example 10.9. (Gauss curvature in an orthogonal parametrization).

In an orthogonal parametrization ($F = 0$) we have

$$K = -\frac{1}{2\sqrt{EG}}\Big(\Big(\frac{E_v}{\sqrt{EG}}\Big)_v + \Big(\frac{G_u}{\sqrt{EG}}\Big)_u\Big)$$

Example 10.10. (Flat torus in $\mathbb{R}^4$).

Let $T = S^1 \times S^1 \subset \mathbb{R}^4$ be the so-called *flat torus*. We have a standard parametrization

$$\boldsymbol{x}(u,v) = (\cos u, \sin u, \cos v, \sin v), \qquad (u,v) \in U$$

with $U = (0, 2\pi) \times (0, 2\pi)$ (and other suitable sets to cover all of S).

We have

$$\boldsymbol{x}_u = (-\sin u, \cos u, 0, 0) \qquad \text{and} \qquad \boldsymbol{x}_v = (0, 0, -\sin v, \cos v),$$

so that $E = G = 1$ and $F = 0$.

Therefore the Gauss curvature is

$$K = -\frac{1}{2\sqrt{EG}}\Big(\Big(\frac{E_v}{\sqrt{EG}}\Big)_v + \Big(\frac{G_u}{\sqrt{EG}}\Big)_u\Big) = 0.$$

Example 10.11. (Surfaces with the same Gauss curvature are not necessarily isometric).

Let $U = (0, 2\pi) \times (0, \infty)$ and let S, $\widetilde{S}$ be the surfaces defined by $S = \boldsymbol{x}(U)$, $\widetilde{S} = \boldsymbol{y}(U)$, where $\boldsymbol{x}, \boldsymbol{y} \colon U \longrightarrow \mathbb{R}^3$ are defined by

$$\boldsymbol{x}(u,v) = (v\cos u, v\sin u, u), \quad \boldsymbol{y}(u,v) = (v\cos u, v\sin u, \log v), \quad (u,v) \in U.$$

(thus S is an open subset of the helicoid and $\widetilde{S}$ is an open subset of a surface of revolution).

The coefficients of the first fundamental forms of S resp. $\widetilde{S}$ w.r.t. $\boldsymbol{x}$ resp. $\boldsymbol{y}$ are

$$E = v^2 + 1, \quad F = 0, \quad G = 1 \qquad \text{and} \qquad \widetilde{E} = v^2, \quad \widetilde{F} = 0, \quad \widetilde{G} = 1 + \frac{1}{v^2}.$$

Calculating the Gauss curvature for S and $\widetilde{S}$ gives

$$K(\boldsymbol{x}(u,v)) = \widetilde{K}(\boldsymbol{y}(u,v)) = -\frac{1}{(v^2+1)^2},$$

and hence $K(p) = \widetilde{K}(f(p))$.

Since the coefficients of the first fundamental form S and $\widetilde{S}$ are different, f cannot be a local isometry (note that $f \circ \boldsymbol{x} = \boldsymbol{y}$, so that $(f \circ \boldsymbol{x})_u \cdot (f \circ \boldsymbol{x})_u = \boldsymbol{y}_u \cdot \boldsymbol{y}_u = \widetilde{E}$ etc.), so since $E \neq \widetilde{E}$, f cannot be an isometry by Proposition 8.15.

11 Curves on surfaces

11.1 Coordinate curves

Definition 11.1. Let S be a regular surface in $\mathbb{R}^n$. A *curve on the surface* S is a smooth map $\boldsymbol{\alpha} \colon I \longrightarrow S$ ($I \subset \mathbb{R}$ is an interval).

Remark 11.2. Recall: If $\boldsymbol{x}\colon U \longrightarrow S$ is a local parametrisation ($U \subset \mathbb{R}^2$ open) and $\boldsymbol{\alpha}\colon I \longrightarrow \boldsymbol{x}(U)$ a curve in $\boldsymbol{x}(U) \subset U$, then we can write

$$\boldsymbol{\alpha}(s) = \boldsymbol{x}(u(s), v(s)),$$

and

$$\boldsymbol{\alpha}' = u'\boldsymbol{x}_u + v'\boldsymbol{x}_v,$$

which implies

$$\|\boldsymbol{\alpha}'(t)\| = \sqrt{E(u(t), v(t))u'(t)^2 + 2F(u(t), v(t))u'(t)v'(t) + \dots}$$

Example 11.3. Coordinate curves: Let $\boldsymbol{x}\colon U \longrightarrow S$ be a local parametrization ($U \subset \mathbb{R}^2$ open) and $(u_0, v_0) \in U$, then

$$u \mapsto \boldsymbol{x}(u, v_0)$$
$$v \mapsto \boldsymbol{x}(u_0, v)$$

are called *coordinate curves* through $p = \boldsymbol{x}(u_0, v_0)$. The local parametrization is given by $(u(s), v(s)) = (s, v_0)$ for the first, and $(u(s), v(s)) = (u_0, s)$ for the second.

One should note that coordinate curves are not intrinsic, they depend on the parametrization.

11.2 Geodesic and normal curvature

Assume now that $S \subset \mathbb{R}^3$, $\boldsymbol{\alpha}\colon I \longrightarrow S \subset \mathbb{R}^3$ is a unit speed curve. Then $\boldsymbol{\alpha}'(s)$ and $\boldsymbol{\alpha}''(s)$ are orthogonal, and

$$\|\boldsymbol{\alpha}''(s)\| = \kappa(s),$$

where $\kappa(s)$ denotes the *curvature* of $\boldsymbol{\alpha}$ as a space curve.

Denote by $\boldsymbol{N}(\boldsymbol{\alpha}(s))$ the Gauss map of the surface S at $\boldsymbol{\alpha}(s)$. Since $\boldsymbol{\alpha}''$ is orthonormal to $\boldsymbol{\alpha}'$, it lies in the plane spanned by $\boldsymbol{N}$ and $\boldsymbol{N} \times \boldsymbol{\alpha}'$.

Definition 11.4 (Geodesic and normal curvature)**.** If $\boldsymbol{\alpha}\colon I \longrightarrow S$ is a curve on a surface S (with Gauss map $\boldsymbol{N}$) parametrized by arc lenth, then we can write

$$\boldsymbol{\alpha}''(s) = \kappa_{\mathrm{g}}(s)\boldsymbol{N}(\boldsymbol{\alpha}(s)) \times \boldsymbol{\alpha}'(s) + \kappa_{\mathrm{n}}(s)\boldsymbol{N}(\boldsymbol{\alpha}(s)).$$

We call $\kappa_{\mathrm{g}}\colon I \longrightarrow \mathbb{R}$ the *geodesic curvature* and $\kappa_{\mathrm{n}}\colon I \longrightarrow \mathbb{R}$ the *normal curvature* of $\boldsymbol{\alpha}$ in S.

For a curve with an arbitrary parametrization on S the geodesic and normal curvatures are defined to be the same as for its unit speed reparametrization, i.e. if $\boldsymbol{\beta} : J \to S$ is a curve, $\boldsymbol{\alpha} : I \to S$ is a unit speed curve, and $\boldsymbol{\beta}(t(s)) = \boldsymbol{\alpha}(s)$, then $\kappa_{\boldsymbol{\beta},\mathrm{n}}(t(s)) = \kappa_{\boldsymbol{\alpha},\mathrm{n}}(s)$, and $\kappa_{\boldsymbol{\beta},\mathrm{g}}(t(s)) = \kappa_{\boldsymbol{\alpha},\mathrm{g}}(s)$. In other words, normal and geodesic curvatures are invariant under reparametrizations by definition.

Remark 11.5. We have (if $\boldsymbol{\alpha}$ is parametrized by arc length!)

$$\kappa_{\mathrm{n}} = \boldsymbol{\alpha}'' \cdot \boldsymbol{N} \qquad \text{and} \qquad \kappa_{\mathrm{g}} = \boldsymbol{\alpha}'' \cdot (\boldsymbol{N} \times \boldsymbol{\alpha}')$$

Furthermore, recall that the curvature κ of a *space curve* is given by $\kappa = \|\boldsymbol{\alpha}''\|$ (if $\boldsymbol{\alpha}$ is parametrized by arc length), and since $\boldsymbol{N}$ and $\boldsymbol{N} \times \boldsymbol{\alpha}'$ form an orthonormal system, we have by Pythagoras' Theorem

$$\kappa = \|\boldsymbol{\alpha}''\| = \sqrt{\kappa_{\mathrm{g}}^2 + \kappa_{\mathrm{n}}^2}$$

Example 11.6. (a) (Plane).

$S = \{ (u,v,0) \mid (u,v) \in \mathbb{R}^2 \}$, then $\boldsymbol{N} = (0,0,1)$.

Let $\boldsymbol{\alpha}\colon I \longrightarrow S$, $\boldsymbol{\alpha}(s) = (u(s), v(s), 0)$, parametrized by arclength; then $\boldsymbol{\alpha}' = (u', v', 0)$, $\boldsymbol{n} \times \boldsymbol{\alpha}' = (-v', u', 0)$ so that

$$\boldsymbol{\alpha}'' = (u'', v'', 0) = \kappa_{\mathrm{g}}(\boldsymbol{N} \times \boldsymbol{\alpha}') + \kappa_{\mathrm{n}}\boldsymbol{N} = \kappa_{\mathrm{g}}(-v', u', 0) + \kappa_{\mathrm{n}}(0,0,1)$$

so that $\kappa_{\mathrm{n}} = 0$, and, if κ is the curvature of $\boldsymbol{\alpha}$, $\kappa = \kappa_{\mathrm{g}}$ (if $\boldsymbol{\alpha}$ is considered as a plane curve) or $\kappa = |\kappa_{\mathrm{g}}|$ (if $\boldsymbol{\alpha}$ is considered as a space curve).

(b) (Lines on surfaces).

Assume that $\boldsymbol{\alpha}(s) = p + s\boldsymbol{v}$, $\|v\| = 1$, parametrizes a line ($s \in I \subset \mathbb{R}$) and that $\boldsymbol{\alpha}(s) \in S$ for all $s \in I$ for some surface $S \subset \mathbb{R}^3$. Then

$$\boldsymbol{\alpha}' = \boldsymbol{v}, \qquad \boldsymbol{\alpha}'' = (0,0,0),$$

so that $\kappa_{\mathrm{g}} = 0$ and $\kappa_{\mathrm{n}} = 0$, i.e., the geodesic and normal curvature of a line on a surface both vanish.

Theorem 11.7 (Meusnier)**.** All curves $\boldsymbol{\beta}$ through $p \in S$ with the same tangent vector $\boldsymbol{w} \in T_pS$ have the same normal curvature

$$\kappa_{\mathrm{n}}(s) = I\!I_p\Big(\frac{\boldsymbol{w}}{\|\boldsymbol{w}\|}\Big).$$

In particular, the value $\kappa_{\mathrm{n}}(\boldsymbol{w})$ is well defined for any $\boldsymbol{w} \in T_pS$.

Corollary. Let $p \in S$, $\boldsymbol{w} \in T_pS$, and let Π be the plane through p spanned by $\boldsymbol{N}(p)$ and $\boldsymbol{w}$. Then $\kappa_{\mathrm{n}}(\boldsymbol{w}) = \kappa(\Pi \cap S)$, where $\Pi \cap S$ is considered as a plane curve with tangent vector $\boldsymbol{w}$ at p.

Proposition 11.8. (Normal curvature in a local parametrization)

Let S be a surface in $\mathbb{R}^3$, and let E, F, G and L, M, N be the coefficient of the first and second fundamental forms respectively w.r.t. a parametrization $\boldsymbol{x}$. Further, let $\boldsymbol{\alpha}$ be a curve in S (not necessarily parametrized by arc length) with local parametrization $\boldsymbol{\alpha}(s) = \boldsymbol{x}(u(s), v(s))$. Then

$$\kappa_{\mathrm{n}} = I\!I_p\left(\frac{\boldsymbol{\alpha}'}{\|\boldsymbol{\alpha}'\|}\right) = \frac{(u')^2L + 2u'v'M + (v')^2N}{(u')^2E + 2u'v'F + (v')^2G} = \frac{I\!I_p(\boldsymbol{\alpha}')}{I_p(\boldsymbol{\alpha}')}$$

Proposition 11.9. Let $\boldsymbol{\beta}\colon I \longrightarrow S$ be a curve not necessarily parametrized by arc length, and let $\boldsymbol{N}$ be the Gauss map of S. Then the geodesic curvature of $\boldsymbol{\beta}$ can be calculated as

$$\kappa_{\mathrm{g}} = \frac{1}{\|\boldsymbol{\beta}'\|^3}(\boldsymbol{\beta}' \times \boldsymbol{\beta}'') \cdot \boldsymbol{N}.$$

Definition 11.10. (Asymptotic curves) A curve $\boldsymbol{\alpha}$ on a surface $S \subset \mathbb{R}^3$ is called an *asymptotic curve* if its normal curvature vanishes identically (i.e., if $\kappa_{\mathrm{n}} = 0$).

Remark 11.11. (i) The following are equivalent (TFAE):

(a) $\boldsymbol{\alpha}$ is an asymptotic curve;

(b) $\boldsymbol{\alpha}'' \cdot (\boldsymbol{N} \circ \boldsymbol{\alpha}) = 0$ (if $\boldsymbol{N}$ is the Gauss map of S and $\boldsymbol{\alpha}$ is parametrized by arc length);

(c) $\kappa_{\mathrm{n}} = 0$;

(d) $I\!I_{\boldsymbol{\alpha}(s)}(\boldsymbol{\alpha}'(s)) = 0$ for all s ($\boldsymbol{\alpha}$ not necessarily parametrized by arc length);

(e) $(u')^2L + 2u'v'M + (v')^2N = 0$ in a local parametrization $s \mapsto \boldsymbol{x}(u(s), v(s))$ of $\boldsymbol{\alpha}$.

In particular, $I\!I_p$ is not positive or negative definite along $\boldsymbol{\alpha}$, so $\boldsymbol{\alpha}$ has to be in the *hyperbolic* or *flat* region of the surface.

(ii) $\kappa_n(\boldsymbol{w}) = 0$ for $\boldsymbol{w} \in T_pS$ implies $K(p) \leq 0$.

(iii) If $\boldsymbol{\alpha}$ is a line on S, then $\kappa_n = 0$, i.e., any line on a surface is an asymptotic curve.

Example 11.12. (Asymptotic curves on a surface of revolution/catenoid)

Recall that on a surface of revolution obtained by rotating a curve $\boldsymbol{\alpha}$ given by $\boldsymbol{\alpha}(v) = (f(v), 0, g(v))$ around the z-axis, we have

$$L = \frac{-fg'}{\|\boldsymbol{\alpha}'\|}, \quad M = 0, \quad N = \frac{f''g' - f'g''}{\|\boldsymbol{\alpha}'\|}$$

(see Example 9.13). A curve $\boldsymbol{\beta}$ parametrized locally by $\boldsymbol{\beta}(t) = \boldsymbol{x}(u(t), v(t))$ is an asymptotic curve iff $(u')^2L + 2u'v'M + (v')^2N = 0$, i.e., iff

$$(u')^2 fg' = (v')^2(f''g' - f'g'')$$

If in particular, $f(v) = \cosh v$ and $g(v) = v$ (i.e., the surface of revolution is a *catenoid*), then the above equation becomes

$$(u')^2 \cosh v = (v')^2 \cosh v, \quad \text{or,} \quad u' = \pm v', \quad \text{i.e.,} \quad u = \pm v + c$$

for some constant $c \in \mathbb{R}$.

11.3 Lines of curvature

Definition 11.13. (Lines of curvature)

A curve $\boldsymbol{\alpha}\colon I \longrightarrow S$ on a surface S in $\mathbb{R}^3$ is called a *line of curvature* if $\boldsymbol{\alpha}'(s)$ is a principal direction at $\boldsymbol{\alpha}(s)$ for all $s \in I$, i.e., $\boldsymbol{\alpha}'(s)$ is an eigenvector of the Weingarten map at $\boldsymbol{\alpha}(s)$ for all s.

Equivalently, $\boldsymbol{\alpha}$ is a line of curvature if there is a function $\lambda\colon I \longrightarrow \mathbb{R}$ such that

$$-d\boldsymbol{N}_{\boldsymbol{\alpha}(s)}(\boldsymbol{\alpha}'(s)) = \lambda(s)\boldsymbol{\alpha}'(s)$$

for all $s \in I$. (Here $\lambda(s)$ is a principal curvature at $\boldsymbol{\alpha}(s)$.)

Remark 11.14. Note that if the eigenvalues of a symmetric 2×2-matrix are different, then the corresponding eigenvectors are orthogonal. Hence, each non-umbilic point ($\kappa_1(p) \neq \kappa_2(p)$) has two lines of curvature through it, and they intersect orthogonally. In an umbilic point, this family of orthogonally intersecting curves has a singularity.

Moreover any direction at an umbilic point is principal. In particular, on a sphere ($\kappa_1 = \kappa_2 > 0$) or a plane ($\kappa_1 = \kappa_2 = 0$) any curve is a line of curvature.

Proposition 11.15. (Lines of curvature in a local parametrisation) Let E, F, G and L, M, N be the coefficients of the first and second fundamental forms respectively w.r.t. a local parametrization $\boldsymbol{x}\colon U \longrightarrow S$, and let $\boldsymbol{\alpha}$ be a curve in S with local parametrization $\boldsymbol{\alpha}(s) = \boldsymbol{x}(u(s), v(s))$. Then $\boldsymbol{\alpha}$ is a line of curvature if and only if

$$\det \begin{pmatrix} (v')^2 & -u'v' & (u')^2 \\ E & F & G \\ L & M & N \end{pmatrix} = 0$$

or, equivalently,

$$(FN - GM)(v')^2 + (EN - GL)u'v' + (EM - FL)(u')^2 = 0.$$

Example 11.16. (Hyperbolic paraboloid)

Let $S = \{ (x, y, z) \,|\, xy = z \}$ be a hyperbolic paraboloid parametrized by $\boldsymbol{x}(u, v) = (u, v, uv)$. Then

$$\boldsymbol{x}_u = (1, 0, v), \quad \boldsymbol{x}_v = (0, 1, u), \quad \boldsymbol{N} = D^{-1}(-v, -u, 1), \quad D = (u^2 + v^2 + 1)^{1/2}$$
$$\boldsymbol{x}_{uu} = (0, 0, 0), \quad \boldsymbol{x}_{uv} = (0, 0, 1), \quad \boldsymbol{x}_{vv} = (0, 0, 0)$$

and

$$E = \boldsymbol{x}_u \cdot \boldsymbol{x}_u = 1 + v^2, \quad F = \boldsymbol{x}_u \cdot \boldsymbol{x}_v = uv, \quad G = \boldsymbol{x}_v \cdot \boldsymbol{x}_v = 1 + u^2,$$
$$L = \boldsymbol{x}_{uu} \cdot \boldsymbol{N} = 0, \quad M = \boldsymbol{x}_{uv} \cdot \boldsymbol{N} = 1/D, \quad N = \boldsymbol{x}_{vv} \cdot \boldsymbol{N} = 0$$

Therefore, $\boldsymbol{\alpha}$ with $\boldsymbol{\alpha}(s) = \boldsymbol{x}(u(s), v(s))$ is a *line of curvature* iff

$$\det \begin{pmatrix} (v')^2 & -u'v' & (u')^2 \\ 1 + v^2 & uv & 1 + u^2 \\ 0 & 1/D & 0 \end{pmatrix} = (u')^2(1 + v^2)/D - (v')^2(1 + u^2)/D = 0,$$

which is equivalent to

$$\frac{u'}{(1 + u^2)^{1/2}} = \pm \frac{v'}{(1 + v^2)^{1/2}},$$

and after integrating,

$$\operatorname{arcsinh} u = \pm \operatorname{arcsinh} v + c$$

for some constant $c \in \mathbb{R}$. For example, if $c = 0$, then $u = \pm v$, or $s \mapsto \boldsymbol{x}(s, \pm s) = (s, \pm s, \pm s^2)$ are the lines of curvature through $p = (0, 0, 0)$.

The *asymptotic curves* here are given by

$$(u')^2 L + 2u'v'M + (v')^2 M = 2u'v'/D = 0,$$

i.e., $u' = 0$ or $v' = 0$, so the asymptotic curves are the coordinate curves $s \mapsto \boldsymbol{x}(s, v_0)$ or $s \mapsto \boldsymbol{x}(u_0, s)$

Remark 11.17. (a) On a *line of curvature*, the *normal curvature* is a *principal curvature*.

Indeed, since $\boldsymbol{\alpha}$ is a line of curvature, we have $-d_{\boldsymbol{\alpha}(s)}\boldsymbol{N}(\boldsymbol{\alpha}'(s)) = \lambda(s)\boldsymbol{\alpha}'(s)$, and $\lambda(s)$ is a principal curvature at $\boldsymbol{\alpha}(s)$.

On the other hand,

$$\kappa_{\mathrm{n}}(s) = \frac{I\!I_{\boldsymbol{\alpha}(s)}(\boldsymbol{\alpha}'(s))}{I_{\boldsymbol{\alpha}(s)}(\boldsymbol{\alpha}'(s)} = \frac{\langle -d_{\boldsymbol{\alpha}(s)}\boldsymbol{N}(\boldsymbol{\alpha}'(s)), \boldsymbol{\alpha}'(s)\rangle}{\langle \boldsymbol{\alpha}'(s), \boldsymbol{\alpha}'(s)\rangle} = \frac{\langle \lambda(s)\boldsymbol{\alpha}'(s), \boldsymbol{\alpha}'(s)\rangle}{\langle \boldsymbol{\alpha}'(s), \boldsymbol{\alpha}'(s)\rangle} = \lambda(s)$$

(b) Assume that a line $\boldsymbol{\alpha}$ (or a part of it) belongs to a surface. When is this line a *line of curvature*?

On a line, the normal curvature is 0, hence by the first part, one of its principal curvatures, say κ_1, has to vanish on $\boldsymbol{\alpha}$. But this means that the Gauss curvature (as the product of the two principal curvatures $K = \kappa_1\kappa_2$) has to vanish (and vice versa). Hence if $\boldsymbol{\alpha}\colon I \longrightarrow S$ is a line in S, then

$$\boldsymbol{\alpha} \text{ is a line of curvature} \quad \Leftrightarrow \quad \big(K(\boldsymbol{\alpha}(s)) = 0 \quad \forall s \in I\big).$$

This is equivalent to $LN - M^2 = 0$.

Proposition 11.18. (Lines of curvature for a principal parametrization)

If $\boldsymbol{x}$ is a principal parametrization of a surface $S \subset \mathbb{R}^3$ (i.e., $F = 0$ and $M = 0$), then the coordinate curves are lines of curvature.

Example 11.19. (Lines of curvature for a surface of revolution)

On a surface of revolution, the coordinate curves of the standard parametrization given by $\boldsymbol{x}(u,v) = (f(v)\cos u, f(v)\sin u, g(v))$ are also lines of curvature.

Remark 11.20. Note that the converse of Proposition 11.18 is also true in the following sense: if a parametrization $\boldsymbol{x}$ is principal and the umbilic points are isolated, then the lines of curvature are coordinate curves.

12 Geodesics

Definition 12.1. Let $\boldsymbol{\alpha}\colon I \longrightarrow S$ be a (regular) curve on a surface $S \subset \mathbb{R}^3$. $\boldsymbol{\alpha}$ is called *geodesic* if $\boldsymbol{\alpha}''$ is normal to S (i.e., $\boldsymbol{\alpha}''(s)$ is orthogonal to $T_{\boldsymbol{\alpha}(s)}S$ for all $s \in I$).

Note that the curve does not need to be parametrized by arc length, but we have:

Proposition 12.2 (Geodesics have constant speed)**.** Let $\boldsymbol{\alpha}$ be a geodesic, then $\|\boldsymbol{\alpha}'\|$ is constant, i.e., there exists $c > 0$ such that $\boldsymbol{\alpha}'(s) = c$ for all $s \in I$.

In other words, a geodesic is always parametrized *proportionally* to arc length.

Example 12.3.

(a) **Lines are geodesics.**

Let S be a surface and $\boldsymbol{\alpha}$ be a line in S. Then $\boldsymbol{\alpha}''(s) = 0$, hence $\boldsymbol{\alpha}''$ is normal to any vector (in particular to the tangent plane $T_{\boldsymbol{\alpha}(s)}S$). Therefore, $\boldsymbol{\alpha}$ is a geodesic.

(b) **Geodesics on a cylinder.**

Let $S = \{\,(x,y,z) \mid x^2 + y^2 = 1\,\}$, then any geodesic $\boldsymbol{\alpha}$ on S is parametrized by

$$\boldsymbol{\alpha}(s) = (\cos(as+b), \sin(as+b), \lambda s + \mu)$$

for some $\lambda, \mu, a, b \in \mathbb{R}$. If $a = 0$ then $\boldsymbol{\alpha}$ is a meridian, if $\lambda = 0$ then $\boldsymbol{\alpha}$ is a parallel.

(c) **Great circles on a sphere are geodesics.**

A *great circle* on a sphere is the curve given by the intersection of the sphere with a plane through its origin.

Let $S = \{\,(x,y,z) \mid x^2 + y^2 + z^2 = 1\,\}$, and let $\boldsymbol{v}, \boldsymbol{w}$ be orthonormal in $\mathbb{R}^3$. Set

$$\boldsymbol{\alpha}(s) = (\cos s)\boldsymbol{v} + (\sin s)\boldsymbol{w}$$

for $s \in I$ (I some interval). Then $\boldsymbol{\alpha}''(s) = -\boldsymbol{\alpha}(s) = -\boldsymbol{N}(\boldsymbol{\alpha}(s))$, hence $\boldsymbol{\alpha}$ is orthogonal to $T_{\boldsymbol{\alpha}(s)}S$ and $\boldsymbol{\alpha}$ is a geodesic.

Proposition 12.4 (Equivalent characterization of geodesics)**.** The following are equivalent (TFAE):

(a) $\boldsymbol{\alpha}$ is a geodesic;

(b) $\boldsymbol{\alpha}$ has constant speed and its geodesic curvature vanishes.

Proposition 12.5 (Geodesics in a local parametrization)**.** Let $\boldsymbol{\alpha}\colon I \longrightarrow S$ be a curve on a surface $S \subset \mathbb{R}^3$, and let $\boldsymbol{x}\colon U \longrightarrow S$ be a local parametrization. We write $\boldsymbol{\alpha}(s) = \boldsymbol{x}(u(s), v(s))$ and E, F, G for the coefficients of the first fundamental form w.r.t. $\boldsymbol{x}$. Then the following are equivalent:

(a) $\boldsymbol{\alpha}$ is a geodesic;

(b) $\boldsymbol{\alpha}'' \cdot \boldsymbol{x}_u = 0$ and $\boldsymbol{\alpha}'' \cdot \boldsymbol{x}_v = 0$;

(c)

$$u''E + \frac{1}{2}(u')^2 E_u + u'v'E_v + (v')^2\Big(F_v - \frac{1}{2}G_u\Big) + v''F = 0,$$
$$v''G + \frac{1}{2}(v')^2 G_v + u'v'G_u + (u')^2\Big(F_u - \frac{1}{2}E_v\Big) + u''F = 0.$$

Let us now state the main theorem about geodesics:

Theorem 12.6 (Local existence and uniqueness of geodesics)**.** (a) Let $p \in S$, $\boldsymbol{w} \in T_pS \setminus \{0\}$. Then there exists $\varepsilon > 0$ and a *unique* geodesic $\boldsymbol{\alpha} \colon (-\varepsilon, \varepsilon) \longrightarrow S$ such that $\boldsymbol{\alpha}(0) = p$ and $\boldsymbol{\alpha}'(0) = \boldsymbol{w}$.

(b) Geodesics are determined entirely by the coefficients of the first fundamental form E, F and G (and their derivatives) in a local parametrization.

Corollary 12.7 (Isometries take geodesics to geodesics)**.** Let $f \colon S \longrightarrow \widetilde{S}$ be a local isometry between two surfaces S and $\widetilde{S}$, and let $\boldsymbol{\alpha} \colon I \longrightarrow S$ be a geodesic on S. Then $f \circ \boldsymbol{\alpha} \colon I \longrightarrow \widetilde{S}$ is also a geodesic on $\widetilde{S}$.

Example 12.8.

(a) **Plane.**

We know that $E = G = 1$ and $F = 0$ (in the standard parametrization $(u, v) \in \mathbb{R}^2$), so the local equation for a geodesic is

$$u'' = 0 \qquad \text{and} \qquad v'' = 0$$

This means that

$$u(s) = u_0 + as \qquad \text{and} \qquad v(s) = v_0 + bs$$

for some numbers u_0, v_0, a, b ((u_0, v_0) is the starting point and $\boldsymbol{w} = (a, b)$ is the initial speed vector). These are all geodesics on a plane

(b) **Cylinder.**

Let $S := \{\,(x, y, z) \mid x^2 + y^2 = 1\,\}$ be a cylinder and $f \colon \mathbb{R}^2 \longrightarrow S$ be given by $f(u, v) = (\cos u, \sin u, v)$, then f is a local isometry. Geodesics on the cylinder S are just images of lines under f:

- lines $s \mapsto (\cos u_0, \sin u_0, s)$ (u_0 some constant): image of the line $s \mapsto (u_0, s)$;
- circles $s \mapsto (\cos s, \sin s, v_0)$ (v_0 some constant): image of the line $s \mapsto (s, v_0)$;
- helices $s \mapsto (\cos s, \sin s, v_0 + as)$ (v_0, a some constants): image of the line $s \mapsto (s, v_0 + as)$ (the circles above are the case $a = 0$)

These are all geodesics (use the local *uniqueness* result of Theorem 12.6), cf. Example 12.3.

Remark 12.9 (Minimizing property of geodesics)**.** (a) The shortest curve between two points on a surface is a geodesic (if parametrized proportionally to arc length).

(b) Converse is false: not all geodesics connecting two points minimize the distance.

(c) A minimizing curve (a geodesic) might not be unique. Moreover, there might be infinitely many of these.

(d) There might be no geodesic joining two points on a surface.

Example 12.10 (Geodesics on a surface of revolution)**.** Let S be a surface of revolution with local parametrization

$$\boldsymbol{x}(u,v) = (f(v)\cos u, f(v)\sin u, g(v)),$$

and let $\boldsymbol{\alpha}(s) = \boldsymbol{x}(u(s), v(s))$ be a geodesic on S. Then the equations from Prop. 12.5 reduce to

$$u''E + u'v'E_v = 0,$$
$$v''G + \frac{1}{2}v'^2 G_v - \frac{1}{2}u'^2 E_v = 0.$$

The first equation is equivalent to $(u'E)' = 0$, or

$$u' = \frac{c}{f^2}$$

for some constant $c \in \mathbb{R}$.

Assuming that the the generating curve $(f, 0, g)$ is unit speed, the second equation is reduced to $v''G - u'^2 E_v/2 = 0$, or, equivalently,

$$v'' - u'^2 f f' = 0$$

as $E = f^2$.

Corollary. (a) All meridians are geodesics

(b) A parallel $v = v_0$ is geodesic if and only if $f'(v_0) = 0$.

Proposition 12.11 (Clairaut relation)**.** Let S be a surface of revolution with local parametrization

$$\boldsymbol{x}(u,v) = (f(v)\cos u, f(v)\sin u, g(v)),$$

and let $\boldsymbol{\alpha}(s) = \boldsymbol{x}(u(s), v(s))$ be a geodesic on S. Denote by $\Theta(s)$ the angle formed formed by $\boldsymbol{\alpha}'(s)$ and the parallel through $\boldsymbol{\alpha}(s)$. Then

$$f(v(s))\cos\Theta(s) = \text{const}$$

Example 12.12 (Torus of revolution)**.** Let S be a torus of revolution with local parametrization

$$\boldsymbol{x}(u,v) = ((R + r\cos v)\cos u, (R + r\cos v)\sin u, r\sin v)$$

for $0 < r < R$. Let $\boldsymbol{\alpha}(s)$ be a geodesic on S through a point $\boldsymbol{\alpha}(0) = (R + r, 0, 0)$. Denote by Θ_0 the angle formed by $\boldsymbol{\alpha}'(0)$ and $\boldsymbol{x}_u$. Then $\boldsymbol{\alpha}(s)$ satisfies the equation

$$(R + r\cos v(s))\cos\Theta(s) = (R + r)\cos\Theta_0$$

Definition 12.13. A geodesic $\boldsymbol{\alpha}\colon I \longrightarrow S$ is *closed* if there is $c \in \mathbb{R}_+$ such that $\boldsymbol{\alpha}(s + c) = \boldsymbol{\alpha}(s)$ for every $s \in I$.

Example 12.14. (a) Every geodesic on a sphere is closed.

(b) The only closed geodesics on a cylinder are parallels.

Example 12.15. There are no closed geodesics on an elliptic paraboloid of revolution.

13 Gauss–Bonnet theorems

13.1 A bit of topology

Definition 13.1. (a) A surface $S \subset \mathbb{R}^n$ is a *closed surface* if S is bounded, connected and closed (as a set).

(b) A surface is *oriented* if the Gauss map can be defined globally as a continuous map.

(c) A *region* of a surface S is a subset of S such that its boundary consists of a finite number of smooth curves (called *edges*) and its interior is non-empty. We call the points in which two smooth curves meet on the boundary *vertices* (and we assume for simplicity that the curves meet non-tangentially).

(d) A *triangle* is a region with three edges and three vertices homeomorphic to a disc (note that the edges, as well as the vertices, may coincide).

(e) A *triangulation* of a (bounded) region R is a subdivision of S into a finite number of triangles meeting only in common edges or common vertices.

(f) The *Euler characteristic* of a region R is defined by

$$\begin{aligned}\chi(R) :&= F(R) - E(R) + V(R)\\ &= \#\text{triangles} - \#\text{edges} + \#\text{vertices},\end{aligned}$$

where $F(R)$ is the number of triangles, $E(R)$ the number of edges and $V(R)$ the number of vertices of the triangulation.

Example 13.2. A closed disc has Euler characteristic 1, a sphere has Euler characteristic 2, a closed cylinder $S^1 \times [0,1]$ (as well as a torus) has Euler characterisic 0.

A priori, the Euler characteristic may depend on the triangulation.

Theorem 13.3. The Euler characteristic is independent of the triangulation.

Basically, oriented closed surfaces can be topologically characterized by their Euler characteristic:

$$\chi(S) = 2 - 2g,$$

where g is the *genus* of S (roughly, the number of "handles" in S).

Theorem 13.4 (Jordan Curve Theorem)**.** Let S be a surface homeomorphic to the plane, and let $\boldsymbol{\alpha}\colon [0,1] \longrightarrow S$ be a simple closed curve (i.e., $\boldsymbol{\alpha}(0) = \boldsymbol{\alpha}(1)$ and $\boldsymbol{\alpha}(t_1) \neq \boldsymbol{\alpha}(t_2)$ for $t_1 < t_2$ other than $t_1 = 0$, $t_2 = 1$). Then $S \setminus \boldsymbol{\alpha}(I)$ has exactly two components, and one of them is homeomorphic to a disc.

13.2 The Gauss–Bonnet theorem

Definition 13.5. Let $R \subset S$ be a region.

(a) Denote by $\mathrm{d}A$ the area measure of a surface S (locally, $\mathrm{d}A = \sqrt{EG - F^2}\,\mathrm{d}u\,\mathrm{d}v$), and we will write

$$\int_R K\,\mathrm{d}A$$

for the integral of the Gauss curvature over R (the *total* Gauss curvature of R).

(b) Denote by $\mathrm{d}s$ the length measure of a curve or the boundary of a region, we will write

$$\int_{\partial R} \kappa_{\mathrm{g}}\,\mathrm{d}s = \sum_{j=1}^{r} \int_{I_j} \kappa_{\mathrm{g},\boldsymbol{\alpha}_j}(s)\,\mathrm{d}s_j$$

for the line integral of the geodesic curvature along the boundary of a region consisting of r smooth curves $\boldsymbol{\alpha}_j$.

(c) Let us parametrize the curves along ∂R counter-clockwise, and the curves are numbered in the same direction. We define the *angle* ϑ_j at the vertex v_j (where curve $\boldsymbol{\alpha}_{j-1}$ and $\boldsymbol{\alpha}_j$ meet) as the angle between the tangent vector of $\boldsymbol{\alpha}_{j-1}$ with the tangent vector of $\boldsymbol{\alpha}_j$, i.e. ϑ_j is the exterior angle of R at v_j.

Note that all objects here are intrinsic (Gauss curvature, geodesic curvature), so we can state the Gauss–Bonnet Theorem for any surface S embedded in $\mathbb{R}^n$ (not only for $n = 3$).

Theorem 13.6 (Global Gauss–Bonnet Theorem)**.** Let R be a region in an oriented surface S. Then

$$\int_R K\,\mathrm{d}A + \int_{\partial R} \kappa_{\mathrm{g}}\,\mathrm{d}s + \sum_{j=1}^{r} \vartheta_j = 2\pi\chi(R).$$

Let us mention some special cases.

Corollary 13.7 (Special cases of the Gauss–Bonnet Theorem)**.**

(a) *(R bounded by geodesics)* If the region R is bounded piecewise by *geodesics*, then

$$\int_R K\,\mathrm{d}A + \sum_{j=1}^{r} \vartheta_j = 2\pi\chi(R).$$

(b) *(R bounded by a closed geodesic)* If γ is a simple closed geodesic and R is a region having γ as its boundary, then

$$\int_R K\,\mathrm{d}A = 2\pi\chi(R).$$

(c) *(No boundary, case $R = S$, $\partial R = \emptyset$)* If S is a closed surface, then

$$\int_S K\,\mathrm{d}A = 2\pi\chi(S).$$

Theorem 13.8 (Local Gauss–Bonnet Theorem/Gauss–Bonnet Theorem for triangles)**.** Let T be a triangle in an oriented surface S with interior angles α, β and γ. Then

$$\int_T K\,\mathrm{d}A + \int_{\partial T} \kappa_{\mathrm{g}}\,\mathrm{d}s = \alpha + \beta + \gamma - \pi.$$

Some more special cases.

Corollary 13.9. Assume that S is a surface of constant Gauss curvature K. Assume additionally, that T is a geodesic triangle in S (i.e., ∂T consists of three arcs of geodesics). Then

$$K \cdot (\operatorname{area} T) = \alpha + \beta + \gamma - \pi.$$

Example 13.10.

(a) On a sphere ($K = 1$), the sum of angles in a (geodesic) triangle is always *larger* than π and the difference is equal to the area of the triangle.

(b) On a plane ($K = 0$), the sum of angles in a (geodesic) triangle is always π (independent of the area of the triangle).

(c) On the hyperbolic plane ($K = -1$), the sum of angles in a (geodesic) triangle is always *smaller* than π and the difference is equal to the area of the triangle.

Example 13.11. (a) The total Gauss curvature of the region R of a unit sphere given by the triangle with vertices at the North pole and two points on the equator at distance one quarter of the circumference is equal to $\pi/2$ as R covers one eighth of the surface of the unit sphere. On the other hand, one can observe that R is a regular right-angled triangle, so the statement of the local Gauss–Bonnet theorem becomes "area of $R = 3\pi/2 - \pi$".

(b) The total Gauss curvature of a surface T homeomorphic to a torus is equal to zero since the Euler characteristic is zero. In particular, if T is not flat everywhere, then it contains elliptic, parabolic and flat points.

Example 13.12. Let S be homeomorphic to the plane $\mathbb{R}^2$, and assume that $K \leq 0$ everywhere on S. Then S cannot have any simple closed geodesic.

Indeed, by the Jordan curve theorem, a simple closed curve $\boldsymbol{\alpha}$ encloses two regions, one of them homeomorphic to a disc; call this region R. If we assume now that $\boldsymbol{\alpha}$ were a closed geodesic, then its geodesic curvature would vanish and there would be no vertices, hence by the Gauss–Bonnet theorem we would have

$$\int_R K \, \mathrm{d}A + \underbrace{\int_{\partial R} \kappa_{\mathrm{g}} \, \mathrm{d}s}_{=0} + \underbrace{\sum_{j=1}^{r} \vartheta_j}_{=0} = 2\pi \underbrace{\chi(R)}_{=1}$$

as the Euler characteristic of a disc is $\chi(R) = 1$ (the same as for a triangle). But since $K \leq 0$, the integral $\int_R K \, \mathrm{d}A \leq 0$, and this is a contradiction. Therefore, there is no such geodesic.

Example 13.13. One can verify the local Gauss–Bonnet theorem explicitly for an "ideal" triangle on a hyperbolic plane: the area of the region bounded by two vertical lines $u = u_1$ and $u = u_2$ and a semicircle intersecting the real axis at points u_1 and u_2 is equal to π.

Example 13.14. Let T be a flat torus in $\mathbb{R}^4$ (i.e. a torus parametrized by $\boldsymbol{x}(u, v) = (\cos u, \sin u, \cos v, \sin v)$). The Gauss–Bonnet theorem implies that any non-closed geodesic on T is not self-intersecting.

The same result can be obtained by considering the geodesics on T as images of lines on $\mathbb{R}^2$ under local isometry $\boldsymbol{x}$.