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Differential Geometry III, Term 1 (Section 3)

3 Plane curves

3.1 Tangent and normal vectors. Curvature

Let $\boldsymbol{\alpha}\colon I \longrightarrow \mathbb{R}^2$ be a plane curve parametrized by arc length, i.e., $\boldsymbol{\alpha}'(s) = \mathbf{t}(s)$ is a unit vector.

Definition 3.1. The *unit normal vector* $\mathbf{n}(s)$ is the vector obtained by rotating $\mathbf{t}(s)$ anticlockwise through $\pi/2$.

In coordinates, if $\boldsymbol{\alpha}(s) = (x(s), y(s))$, then

$$\mathbf{t}(s) = (x'(s), y'(s)), \qquad \mathbf{n}(s) = (-y'(s), x'(s))$$

Remark. Differentiating the equation $1 = \|\mathbf{t}(s)\|^2 = \mathbf{t}(s) \cdot \mathbf{t}(s)$ gives

$$0 = \mathbf{t}'(s) \cdot \mathbf{t}(s) + \mathbf{t}(s) \cdot \mathbf{t}'(s) = 2\mathbf{t}'(s) \cdot \mathbf{t}(s).$$

In particular, $\mathbf{t}(s)$ and $\mathbf{t}'(s)$ are orthogonal, and hence $\mathbf{t}'(s)$ is parallel to the normal vector $\mathbf{n}(s)$ (which is also orthogonal to $\mathbf{t}(s)$). (Note that we use here the fact that we are in $\mathbb{R}^2$, otherwise the last conclusion that $\mathbf{t}'(s)$ is parallel to $\mathbf{n}(s)$ is not true!)

Definition 3.2. The *(signed) curvature* $\kappa(s)$ of a plane curve $\boldsymbol{\alpha}\colon I \longrightarrow \mathbb{R}^2$ is defined by $\mathbf{t}'(s) = \kappa(s)\mathbf{n}(s)$.

Remark. A way to compute: $\mathbf{n}(s) \cdot \mathbf{t}'(s) = \kappa(s)\mathbf{n}(s) \cdot \mathbf{n}(s) = \kappa(s)$ (since $\mathbf{n}(s)$ is a unit vector), so we have

$$\kappa(s) = \mathbf{n}(s) \cdot \mathbf{t}'(s)$$

If $\boldsymbol{\alpha}$ is given by $\boldsymbol{\alpha}(s) = (x(s), y(s))$, where s is the arc length, then

$$\kappa(s) = -y'(s)x''(s) + x'(s)y''(s),$$

provided the curve is parametrized by arc length.

Example 3.3. (a) *Lines.* $\kappa(s) \equiv 0$.

(b) *Circles.* $\kappa(s) \equiv 1/r$ for a circle of radius r.

Proposition 3.4. Let $\boldsymbol{\alpha}\colon I \longrightarrow \mathbb{R}^2$, $\boldsymbol{\alpha}(u) = (x(u), y(u))$, be a regular curve (not necessarily parametrized by arc length). Then

$$\kappa = \frac{x'y'' - x''y'}{((x')^2 + (y')^2)^{3/2}},$$

where we omitted the argument u of the functions κ, x', x'', y' and y''.

Example. *The ellipse.* Let $\boldsymbol{\alpha}\colon \mathbb{R} \longrightarrow \mathbb{R}^2$, $\boldsymbol{\alpha}(u) = (a\cos u, b\sin u)$ for some constants $a, b > 0$. The curve is regular,

$$\kappa(u) = \frac{ab}{(a^2\sin^2 u + b^2\cos^2 u)^{3/2}}.$$

In particular, the curvature is always positive ($\kappa(u) > 0$ for all $u \in \mathbb{R}$), but not constant if $a \neq b$.

Definition 3.5. Let $\boldsymbol{\alpha}\colon I \longrightarrow \mathbb{R}^2$ be a plane regular curve.

(a) A point $\boldsymbol{\alpha}(u_0)$ is an *inflection point* of $\boldsymbol{\alpha}$ if $\kappa(u) = 0$.

(b) A point $\boldsymbol{\alpha}(u_0)$ is a *vertex* of $\boldsymbol{\alpha}$ if $\kappa'(u) = 0$.

Remark. A vertex is well-defined, i.e. the definition does not depend on the parameter.

Example 3.6. (a) *The cubic.* $\alpha(u) = (u, u^3)$. The only inflection point is $\boldsymbol{\alpha}(0) = (0, 0)$, there are no vertices.

(b) *The parabola.* $\boldsymbol{\alpha}(u) = (u, u^2)$. There are no inflection points, the only vertex is at $u = 0$.

(c) *The ellipse.* There are no inflection points, 4 vertices at $u = k\pi/2$.

Theorem 3.7 (The 4-vertex theorem)**.** Any simple smooth regular closed curve has at least 4 vertices.

Here *simple* means the curve has no self-intersections.

Theorem 3.8 (The fundamental theorem of local theory of plane curves)**.** Given a smooth function $\kappa\colon I \longrightarrow \mathbb{R}$, $s_0 \in I$, $a \in \mathbb{R}^2$ and a unit vector $v_0 \in \mathbb{R}^2$, there is a unique smooth regular curve $\boldsymbol{\alpha}\colon I \longrightarrow \mathbb{R}^2$ parametrized by arc length with curvature $\kappa(s)$ and $\boldsymbol{\alpha}(s_0) = a$, $\boldsymbol{\alpha}'(s_0) = v_0$.

3.2 Evolute and involute of a plane curve

Definition 3.9. Let $\boldsymbol{\alpha}\colon I \longrightarrow \mathbb{R}^2$ be a smooth regular curve parametrized by arc length.

(a) Suppose $\kappa(s) \neq 0$, then

$$\rho(s) = \frac{1}{|\kappa(s)|}$$

is called the *radius of curvature.* The point

$$\boldsymbol{e}(s) = \boldsymbol{\alpha}(s) + \frac{1}{\kappa(s)}\boldsymbol{n}(s)$$

is called the *center of curvature.* Here, $\boldsymbol{n}$ is the unit normal of $\boldsymbol{\alpha}$.

(b) The *evolute (caustic)* of the curve $\boldsymbol{\alpha}$ is the curve traced by the centers of curvature. Thus, a parametrization of the evolutive is

$$\boldsymbol{e}\colon I \longrightarrow \mathbb{R}^2, \qquad \boldsymbol{e}(s) = \boldsymbol{\alpha}(s) + \frac{1}{\kappa(s)}\boldsymbol{n}(s).$$

(c) The *involute* of a plane curve $\boldsymbol{\beta}$ is a curve whose evolute is the initial curve $\boldsymbol{\beta}$.

Remark. Properties of the evolute.
$\boldsymbol{\alpha}$, $\boldsymbol{n}$ and κ are smooth, so $\boldsymbol{e}$ is a smooth curve (whenever $\kappa(s) \neq 0$). Moreover,

$$\boldsymbol{e}'(s) = \boldsymbol{\alpha}'(s) + \frac{1}{\kappa(s)}\boldsymbol{n}'(s) - \frac{\kappa'(s)}{\kappa(s)^2}\boldsymbol{n}(s),$$

which implies

$$\boldsymbol{e}'(s) = -\frac{\kappa'(s)}{\kappa(s)^2}\boldsymbol{n}(s).$$

In particular, we have the following conclutions:

(a) $\boldsymbol{e}'(s)$ is *parallel* to the normal vector $\boldsymbol{n}(s)$ of the original curve $\boldsymbol{\alpha}$.

(b) $\boldsymbol{e}'(s) = \mathbf{0}$ iff $\kappa'(s) = 0$, i.e., the evolute is *singular* at $\boldsymbol{e}(s_0)$ iff $\boldsymbol{\alpha}(s_0)$ is a *vertex*.

(c) The parameter s is *not* an arc length parameter of the evolute $\boldsymbol{e}$: $\|\boldsymbol{e}'(s)\| = |\frac{\kappa'(s)}{\kappa(s)^2}|$ which is not necessarily 1.

Example 3.10. (a) *The ellipse.* $\boldsymbol{\alpha}(u) = (a\cos u, b\sin u)$ for $a > 0$, $b > 0$ and $a \neq b$.

$$\boldsymbol{e}(u) = (a\cos u, b\sin u) + \frac{a^2\sin^2 u + b^2\cos^2 u}{ab}(-b\cos u, -a\sin u).$$

(b) *The circle.* $\boldsymbol{e}(u) =$ the center.