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Differential Geometry III, Term 1 (Section 4)

4 Space curves (curves in $\mathbb{R}^3$)

4.1 The Serret – Frenet formulae

Let $\boldsymbol{\alpha}\colon I \longrightarrow \mathbb{R}^3$ be a smooth regular curve in $\mathbb{R}^3$ parametrized by arc length (i.e., $\boldsymbol{t} = \boldsymbol{\alpha}'$ is the unit tangent vector).

Definition 4.1. The *curvature* $\kappa\colon I \longrightarrow [0, \infty)$ of a space curve $\boldsymbol{\alpha}\colon I \longrightarrow \mathbb{R}^3$ is defined by

$$\kappa(s) := \|\boldsymbol{t}'(s)\|.$$

Remark. The curvature of a *space* curve is always non-negative ($\kappa(s) \geq 0$). For *plane* curves, we introduced the *signed* curvature, which can have negative values. We will see the relation between both concepts later on.

Definition 4.2. Assume that $\kappa(s) > 0$. We define the *principal normal vector* $\boldsymbol{n}(s)$ by

$$\boldsymbol{n}(s) := \frac{1}{\kappa(s)}\boldsymbol{t}'(s).$$

Note that $\boldsymbol{n}(s)$ is really a *unit* vector (and also orthogonal to $\boldsymbol{t}(s)$. We have

$$\boldsymbol{t}'(s) = \kappa(s)\boldsymbol{n}(s).$$

Remark. The *vector product* (or *cross-product*) $\times\colon \mathbb{R}^3 \times \mathbb{R}^3 \longrightarrow \mathbb{R}^3$ in $\mathbb{R}^3$. Recall some facts about the vector product in $\mathbb{R}^3$. Let $\boldsymbol{a}$, $\boldsymbol{b} \in \mathbb{R}^3$.

(a) The *vector product* is defined by

$$\boldsymbol{a} \times \boldsymbol{b} = \begin{pmatrix} a_1 \\ a_2 \\ a_3 \end{pmatrix} \times \begin{pmatrix} a_1 \\ a_2 \\ a_3 \end{pmatrix} = \begin{pmatrix} a_2 b_3 - a_3 b_2 \\ a_3 b_1 - a_1 b_3 \\ a_1 b_2 - a_2 b_1 \end{pmatrix}$$

(b) $\boldsymbol{a} \times \boldsymbol{b}$ is orthogonal to $\boldsymbol{a}$ and $\boldsymbol{b}$, e.g., $(\boldsymbol{a} \times \boldsymbol{b}) \cdot \boldsymbol{a} = 0$.

(c) Antisymmetry: $\boldsymbol{a} \times \boldsymbol{b} = -\boldsymbol{b} \times \boldsymbol{a}$ (in particular, $\boldsymbol{a} \times \boldsymbol{a} = \boldsymbol{0}$).

(d) If $\boldsymbol{a}$ and $\boldsymbol{b}$ are orthogonal unit vectors, then $(\boldsymbol{a}, \boldsymbol{b}, \boldsymbol{a} \times \boldsymbol{b})$ form an orthonormal basis, which is *positively* oriented. Moreover, one has

$$\boldsymbol{b} \times (\boldsymbol{a} \times \boldsymbol{b}) = \boldsymbol{a}, \qquad (\boldsymbol{a} \times \boldsymbol{b}) \times \boldsymbol{a} = \boldsymbol{b}$$

Definition 4.3. The vector $\boldsymbol{b} := \boldsymbol{t} \times \boldsymbol{n}$ is called the *binormal vector* of $\boldsymbol{\alpha}$, and $(\boldsymbol{t}, \boldsymbol{n}, \boldsymbol{b})$ form an orthonormal basis called also *orthonormal frame*.

Since $\boldsymbol{b}'$ is orthogonal to $\boldsymbol{b}$ *and* to $\boldsymbol{t}$, $\boldsymbol{b}'$ is *parallel* to $\boldsymbol{n}$. In particular, the following definition makes sense:

Definition 4.4. The *torsion* $\tau\colon I \longrightarrow \mathbb{R}$ of the space curve $\boldsymbol{\alpha}\colon I \longrightarrow \mathbb{R}^3$ is defined by

$$\boldsymbol{b}'(s) = \tau(s)\boldsymbol{n}(s).$$

Remark. Note that the torsion can have positive or negative values. Moreover, in some books, you will find the equation $\boldsymbol{b}' = -\tau\boldsymbol{n}$ as a definition of the torsion.

Proposition 4.5 (*Serret-Frenet equations*)**.** Let $\boldsymbol{\alpha}\colon I \longrightarrow \mathbb{R}^3$ be a space curve parametrized by arc length with unit tangent, normal and binormal vectors $\boldsymbol{t}$, $\boldsymbol{n}$, $\boldsymbol{b}$. Then

$$\boldsymbol{t}' = \kappa\boldsymbol{n} \tag{4.2}$$

$$\boldsymbol{n}' = -\kappa\boldsymbol{t} - \tau\boldsymbol{b} \tag{4.6}$$

$$\boldsymbol{b}' = \tau\boldsymbol{n} \tag{4.5}$$

or in matrix form

$$\begin{pmatrix} \boldsymbol{t}' \\ \boldsymbol{n}' \\ \boldsymbol{b}' \end{pmatrix} = \begin{pmatrix} 0 & \kappa & 0 \\ -\kappa & 0 & -\tau \\ 0 & \tau & 0 \end{pmatrix} \begin{pmatrix} \boldsymbol{t} \\ \boldsymbol{n} \\ \boldsymbol{b} \end{pmatrix}.$$

Let us now show how to calculate the torsion and curvature for a space curve which is not necessarily parametrized by arc length. This is of practical relevance, since a parametrization is in general not unit speed (i.e., the parameter is not arc length).

Theorem 4.6. Let $\boldsymbol{\alpha}\colon I \longrightarrow \mathbb{R}^3$ be a regular space curve, not necessarily parametrized by arc length. Then the curvature and torsion of $\boldsymbol{\alpha}$ are given by

$$\kappa = \frac{\|\boldsymbol{\alpha}' \times \boldsymbol{\alpha}''\|}{\|\boldsymbol{\alpha}'\|^3} \quad \text{and} \quad \tau = -\frac{(\boldsymbol{\alpha}' \times \boldsymbol{\alpha}'') \cdot \boldsymbol{\alpha}'''}{\|\boldsymbol{\alpha}' \times \boldsymbol{\alpha}''\|^2}$$

(as functions of u), respectively.

Example 4.7. *The helix.* Let $\boldsymbol{\alpha}\colon \mathbb{R} \longrightarrow \mathbb{R}^3$ be given by $\boldsymbol{\alpha}(u) = (a\cos u, a\sin u, u)$ for $a > 0$ (this is a particular case of a helix, see Exercise 4.5). Then $\kappa = \dfrac{a}{a^2+1}$, $\tau(u) = -\dfrac{1}{a^2+1}$.

Remark (Geometric meaning of torsion)**.** The plane through $\boldsymbol{\alpha}(s)$ spanned by $\boldsymbol{t}(s)$ and $\boldsymbol{n}(s)$ is called the *osculating plane.*

The torsion of a curve measures the rate at which the curve pulls away from the osculating plane.

Proposition 4.8 (Exercise)**.** Let $\boldsymbol{\alpha} : I \to \mathbb{R}^3$ be a smooth curve, $\boldsymbol{\alpha}' \times \boldsymbol{\alpha}'' \neq \mathbf{0}$ for $u \in I$. Assume that there is a plane $\Pi \subset \mathbb{R}^3$ containing $\boldsymbol{\alpha}(I)$. Then $\tau(u) \equiv 0$.

We can now express one of the main results on space curve (similar to Theorem 3.8):

Theorem 4.9 (The fundamental theorem of local theory of space curves)**.** Given smooth functions $\kappa\colon I \longrightarrow (0,\infty)$ and $\tau\colon I \longrightarrow \mathbb{R}$, there exists a smooth regular curve $\boldsymbol{\alpha}\colon I \longrightarrow \mathbb{R}^3$ parametrized by arc length such that κ and τ are the curvature and torsion of $\boldsymbol{\alpha}$. Moreover, $\boldsymbol{\alpha}$ is unique up to translations (of the *starting point*) and rotation (of the *starting orthonormal basis*).

Remark 4.10. *Local canonical form of a space curve.* Let $\boldsymbol{\alpha}\colon I \longrightarrow \mathbb{R}^3$ be a space curve parametrized by arc length with $0 \in I$. Then

$$\begin{aligned}\boldsymbol{\alpha}(s) &= \boldsymbol{\alpha}(0) + s\boldsymbol{\alpha}'(0) + \frac{s^2}{2!}\boldsymbol{\alpha}''(0) + \frac{s^3}{3!}\boldsymbol{\alpha}'''(0) + O(s^4) \\ &= \boldsymbol{\alpha}(0) + s\boldsymbol{t}(0) + \frac{s^2}{2!}\underbrace{\boldsymbol{t}'(0)}_{=\kappa(0)\boldsymbol{n}(0)} + \frac{s^3}{3!}\underbrace{\boldsymbol{t}''(0)}_{=\kappa'(0)\boldsymbol{n}(0)+\kappa(0)(-\kappa(0)\boldsymbol{t}(0)-\tau(0)\boldsymbol{b}(0))} + O(s^4)\end{aligned}$$

by the Serret-Frenet formulae. In paricular,

$$\boldsymbol{\alpha}(s) - \boldsymbol{\alpha}(0) = \Big(s - \frac{\kappa(0)^2 s^3}{6}\Big)\boldsymbol{t}(0) + \Big(\frac{\kappa(0)s^2}{2} + \frac{\kappa'(0)s^3}{6}\Big)\boldsymbol{n}(0) - \frac{\kappa(0)\tau(0)s^3}{6}\boldsymbol{b}(0) + O(s^4).$$

If we choose the coordinate system such that $\boldsymbol{t}(0) = (1,0,0)$, $\boldsymbol{n}(0) = (0,1,0)$ and $\boldsymbol{b}(0) = (0,0,1)$, and if we write $\boldsymbol{\alpha}(s) - \boldsymbol{\alpha}(0) = (x(s), y(s), z(s))$, then

$$\begin{aligned} x(s) &= s - \frac{\kappa(0)^2 s^3}{6} \\ y(s) &= \frac{\kappa(0)s^2}{2} + \frac{\kappa'(0)s^3}{6} \\ z(s) &= -\frac{\kappa(0)\tau(0)s^3}{6}. \end{aligned}$$

These equations are called the *local canonical form* of a space curve $\boldsymbol{\alpha}$.