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Differential Geometry III, Term 1 (Section 6)

6 Surfaces

Recall that we defined a curve as a smooth map $\boldsymbol{\alpha}\colon I \longrightarrow \mathbb{R}^n$. So a curve is a deformation of an interval, i.e., a piece of the real line.

Similarly, we look to define a surface as a deformation of an open subset in $\mathbb{R}^2$. Intuitively, a surface in $\mathbb{R}^n$ ($n \geq 3$) is a subset of $\mathbb{R}^n$ that looks locally like a subset of $\mathbb{R}^2$.

6.1 Parametrizations of regular surfaces

Definition 6.1. A subset $S \subset \mathbb{R}^3$ is a *regular surface* if for every point $p \in S$ there exists an open set V in $\mathbb{R}^3$ containing p and a map $\boldsymbol{x}\colon U \longrightarrow S \cap V$, where U is an open subset of $\mathbb{R}^2$, such that

(a) $\boldsymbol{x}$ is a smooth map; that is, if

$$\boldsymbol{x}(u,v) = (x_1(u,v), x_2(u,v), x_3(u,v))$$

then x_1, x_2, x_3 are smooth functions.

(b) $\boldsymbol{x}\colon U \longrightarrow S \cap V$ is a homeomorphism, that is, $\boldsymbol{x}$ has a continuous inverse $\boldsymbol{x}^{-1}\colon S \cap V \longrightarrow U$ *(this condition excludes self-intersections).*

(c) The partial derivatives $\boldsymbol{x}_u$ and $\boldsymbol{x}_v$ are linearly independent for all $(u,v) \in U$ *(this condition excludes singularities and dimension reduction).*

$\boldsymbol{x}$ is called a *local parametrization* of S at p, and $\boldsymbol{x}^{-1}$ is called a *local coordinate chart.*

Let us now come to some main classes of examples of surfaces:

6.2 Graphs of functions and level sets as surfaces

Proposition 6.2. Let $U \subset \mathbb{R}^2$ be open and $g\colon U \longrightarrow \mathbb{R}$ be a smooth function. Then the graph of g,

$$\operatorname{graph}(g) := \left\{ (u, v, g(u,v) \in \mathbb{R}^3 \,\middle|\, (u,v) \in U \right\}$$

is a regular surface in $\mathbb{R}^3$.

Example 6.3.

(a) Let $U = \mathbb{R}^2$ and

$$g(u,v) = \frac{u^2}{a^2} + \frac{v^2}{b^2},$$

then the graph of g is a surface: an *elliptic paraboloid.*

(b) Similarly, let

$$g(u,v) = \frac{u^2}{a^2} - \frac{v^2}{b^2},$$

then the graph of g is a *hyperbolic paraboloid.*

Example 6.4. The *sphere* of radius $r > 0$ and center $\mathbf{0}$ is defined as

$$S(r) := \{ (x, y, z) \in \mathbb{R}^3 \,|\, x^2 + y^2 + z^2 - r^2 = 0 \}.$$

Example 6.5. Consider the function $f\colon \mathbb{R}^3 \longrightarrow \mathbb{R}$ given by $f(x, y, z) = x^2 + y^2 + z^2$. Then the sphere $S(r)$ of radius $r > 0$ is the level set r^2 of f, i.e.,

$$S(r) = \{ (x, y, z) \in \mathbb{R}^3 \,|\, f(x, y, z) = r^2 \} =: f^{-1}(r^2)$$

All the level sets $f^{-1}(r^2)$ are *regular surfaces*, except for $c = r^2 = 0$. The value $c = 0$ corresponds to the point $\boldsymbol{x} = (x, y, z) = \mathbf{0}$. Note that

$$\nabla f = (\partial_x f, \partial_y f, \partial_z) = (2x, 2y, 2z)$$

and that $\nabla f(\boldsymbol{x}) = 0$ iff $\boldsymbol{x} = \mathbf{0}$. We have to exclude such values!

Definition 6.6. Let $U \subset \mathbb{R}^3$ be open and $f\colon U \longrightarrow \mathbb{R}$ be smooth. A value $c \in \mathbb{R}$ in the range $f(U)$ of f is called *regular value* of f if $\nabla f(\boldsymbol{p}) = (\partial_x f, \partial_y f, \partial_z f)(\boldsymbol{p}) \neq \mathbf{0}$ for all $\boldsymbol{p} \in U$ such that $f(\boldsymbol{p}) = c$.
A point $\boldsymbol{p}$ is called *critical point* if $\nabla f(\boldsymbol{p}) = \mathbf{0}$. In this case $c = f(\boldsymbol{p})$ is a *critical value* of f.

So $c = r^2 > 0$ is a regular value of f from the previous example, and $c = 0$ is a critical value.

Proposition 6.7. Let $U \subset \mathbb{R}^3$ be open and $f\colon U \longrightarrow \mathbb{R}$ be smooth, let $c \in f(U)$ be a regular value of f. Then

$$f^{-1}(c) := \{ \boldsymbol{x} \in U \,|\, f(\boldsymbol{x}) = c \}$$

is a regular surface.

Example 6.8.

(a) $S(r) = \{ (x, y, z) \in \mathbb{R}^3 \,|\, x^2 + y^2 + z^2 = r^2 \}$ is the level set of f, where $f(x, y, z) = x^2 + y^2 + z^2$, i.e., $S(r) = f^{-1}(r^2)$. $S(r)$ is a regular surface if $r > 0$.

(b) Let $f\colon \mathbb{R}^3 \longrightarrow \mathbb{R}$ be given by $f(x, y, z) = x^2 + y^2 - z^2$. Let $S = f^{-1}(1)$ be the level set 1 of f. Since $c = 1$ is a regular value of f, S is a regular surface, a *hyperboloid of one sheet.*

(c) With the same f as before, $f^{-1}(-1)$ is called the *hyperboloid of two sheets.* The value -1 is again a regular value, so the hyperboloid of two sheets is regular.

(d) A cylinder given by those points $(x, y, z) \in \mathbb{R}^3$ such that $x^2 + y^2 = 1$ is a regular surface.

6.3 Change of parameters

Definition 6.9. Let U, V be two open sets. A smooth map $\boldsymbol{h}\colon V \longrightarrow U$ is called a *diffeomorphism* if it is bijective and if the inverse $\boldsymbol{h}^{-1}\colon U \longrightarrow V$ is also smooth.

Example 6.10. Let $U = V = \mathbb{R}$. Then $\boldsymbol{h}(x) = x$ is a diffeomorphism, but $\boldsymbol{h}(x) = x^3$ is not.

Proposition 6.11. (a) Let $S \subset \mathbb{R}^3$ be a surface and let $\boldsymbol{x}\colon U \subset \mathbb{R}^2 \longrightarrow S$ be a local parametrization. Let $\boldsymbol{h}\colon V \subset \mathbb{R}^2 \longrightarrow U$ be a diffeomorphism. Then $\boldsymbol{y} = \boldsymbol{x} \circ \boldsymbol{h}\colon V \longrightarrow S$ is also a local parametrization.

(b) Let $\boldsymbol{x}\colon U \longrightarrow S$ and $\boldsymbol{y}\colon V \longrightarrow S$ be two local parametrizations with $\boldsymbol{x}(U) = \boldsymbol{y}(V) \subset S$ (i.e., $\boldsymbol{x}$ and $\boldsymbol{y}$ cover the same region of the surface). Then $\boldsymbol{x}^{-1} \circ \boldsymbol{y}\colon V \longrightarrow U$ is a diffeomorphism.

6.4 Special surfaces

Surfaces constructed by a plane and space curves.

Example 6.12. Surface of revolution. Let I be an open interval in $\mathbb{R}$ and $\tilde{\boldsymbol{\alpha}}\colon I \longrightarrow \mathbb{R}^2$ be a regular smooth plane curve, $\tilde{\boldsymbol{\alpha}}(v) = (f(v), g(v))$. Define a space curve $\boldsymbol{\alpha}(v) = (f(v), 0, g(v))$. Assume that $\boldsymbol{\alpha}$ has no self-intersections (i.e. $\boldsymbol{\alpha}(u) \neq \boldsymbol{\alpha}(v)$ if $u \neq v$) and that $f(v) \neq 0$, so $\boldsymbol{\alpha}$ does not meet the z-axis.

Now rotate $\boldsymbol{\alpha}$ about the z-axis. The set

$$S := \left\{ (f(v)\cos u, f(v)\sin u, g(v)) \,\middle|\, u \in \mathbb{R}, v \in I \right\}$$

is a surface, called a *surface of revolution.*
The curve $\boldsymbol{\alpha}$ is called the *generating curve.* The circles swept out by points of $bm\alpha$ are called *parallels*, and the curves obtained by rotating $\boldsymbol{\alpha}$ through a fixed angle are *meridians.*

Examples: cylinder ($\boldsymbol{\alpha}$ is a vertical line), *catenoid* ($\boldsymbol{\alpha}(v) = (\cosh v, 0, v)$, $v \in \mathbb{R}$).

Example 6.13. Canal surfaces.

Let $\boldsymbol{\alpha}\colon I \longrightarrow \mathbb{R}^3$ be a smooth regular non-self-intersecting space curve parametrized by arc length. Choose $r > 0$ small enough, and consider the family of circles in the normal plane (i.e., spanned by $\boldsymbol{n}(s)$ and $\boldsymbol{b}(s)$ with center $\boldsymbol{\alpha}(s)$ and radius r. These form a surface called a *canal surface* or *tubular neighbourhood* of $\boldsymbol{\alpha}$. This surface is parametrized by

$$\boldsymbol{x}(s, \vartheta) = \boldsymbol{\alpha}(s) + r(\boldsymbol{n}(s)\cos\vartheta + \boldsymbol{b}(s)\sin\vartheta).$$

Example 6.14. Ruled surfaces. Let $\boldsymbol{\alpha}\colon I \longrightarrow \mathbb{R}^3$ be a smooth regular space curve (without self-intersections) and $\boldsymbol{w}\colon I \longrightarrow \mathbb{R}^3$ be a smooth map which is never zero. Suppose that $\boldsymbol{\alpha}'(u)$ is not parallel to $\boldsymbol{w}(u)$ (where $\boldsymbol{w}(u)$ is viewed as a vector). We consider the family of segments of lines through $\boldsymbol{\alpha}(u)$ and parallel to $\boldsymbol{w}(u)$.

These form a surface call a *ruled surface.* If we take $J = (-a, a)$, with a small enough, then

$$\boldsymbol{x}(u, v) = \boldsymbol{\alpha}(u) + v\boldsymbol{w}(u), u \in I, v \in J$$

is a parametrization of a ruled surface.

Example 6.15. $f(x, y, z) := x^2 + y^2 - z^2$ defines a smooth function $f\colon \mathbb{R}^3 \longrightarrow \mathbb{R}$, and 1 is a regular value, hence $S = f^{-1} = \{ (x, y, z) \,|\, x^2 + y^2 - z^2 = 1 \}$ is a regular surface, a *hyperboloid of one sheet.* It is a surface of revolution and a ruled surface.