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Differential Geometry III, Term 2 (Section 10)

10 The Theorema Egregium of Gauss

"Theorema Egregium" means "Remarkable Theorem".

Theorem 10.1 (Theorema Egregium)**.** The Gauss curvature of a surface in $\mathbb{R}^3$ depends on E, F, G and their derivatives only (in a local parametrization).

In other words: the Gauss curvature is *intrinsic*.

Corollary 10.2. A local isometry preserves the Gauss curvature.

The converse is false: a map preserving the Gauss curvature is not necessarily a (local) isometry, see Remark 10.11.

Remark 10.3. Theorem 10.1 does *not* hold for the mean curvature: e.g. $H = 0$ (plane) but $H = 1/(2r)$ (cylinder), although the plane and the cylinder are locally isometric.

Definition 10.4 (Christoffel symbols)**.** Let $\boldsymbol{x}\colon U \longrightarrow S$ be a local parametrization of a surface S in $\mathbb{R}^3$. The Christoffel symbols Γ_{ij}^k $(i, j, k \in \{1, 2\})$ are functions $\Gamma_{ij}^k\colon U \longrightarrow \mathbb{R}$ defined by

$$\begin{aligned}
\boldsymbol{x}_{uu} &= \Gamma_{11}^1 \boldsymbol{x}_u + \Gamma_{11}^2 \boldsymbol{x}_v + L\boldsymbol{N}\\
\boldsymbol{x}_{uv} &= \Gamma_{12}^1 \boldsymbol{x}_u + \Gamma_{12}^2 \boldsymbol{x}_v + M\boldsymbol{N}\\
\boldsymbol{x}_{vu} &= \Gamma_{21}^1 \boldsymbol{x}_u + \Gamma_{21}^2 \boldsymbol{x}_v + M\boldsymbol{N}\\
\boldsymbol{x}_{vv} &= \Gamma_{22}^1 \boldsymbol{x}_u + \Gamma_{22}^2 \boldsymbol{x}_v + N\boldsymbol{N}
\end{aligned}$$

In particular, $\Gamma_{ij}^k = \Gamma_{ji}^k$.

Lemma 10.5.

(a) We have the identities

$$\begin{aligned}
\boldsymbol{x}_{uu}\cdot\boldsymbol{x}_u &= \frac{1}{2}E_u & \boldsymbol{x}_{vv}\cdot\boldsymbol{x}_v &= \frac{1}{2}G_v\\
\boldsymbol{x}_{uv}\cdot\boldsymbol{x}_u &= \frac{1}{2}E_v & \boldsymbol{x}_{uv}\cdot\boldsymbol{x}_v &= \frac{1}{2}G_u\\
\boldsymbol{x}_{vv}\cdot\boldsymbol{x}_u &= F_v - \frac{1}{2}G_u & \boldsymbol{x}_{uu}\cdot\boldsymbol{x}_v &= F_u - \frac{1}{2}E_v
\end{aligned}$$

for the coefficients E, F and G of the first fundamental form with respect to a parametrization $\boldsymbol{x}$.

(b) The Christoffel symbols are uniquely determined by E, F, G and their first derivatives.

Corollary 10.6. Gauss' Theorema Egregium allows us to define the Gauss curvature for *any* surface S just using the *first fundamental form*.

Example 10.7 (Gauss curvature of the hyperbolic plane)**.** Recall that we define the hyperbolic plane as a surface $\mathbb{H}$ parametrized by $x\colon U \longrightarrow H$ with

$$U = \mathbb{R} \times (0, \infty), \qquad E(u,v) = G(u,v) = \frac{1}{v^2}, \quad F(u,v) = 0.$$

Step 1 — Christoffel symbols: We first calculate the Christoffel symbols in the case that $F = 0$ (you can read off Γ_{ij}^k directly):

$$\begin{cases} E\Gamma_{11}^1 &= \frac{1}{2}E_u \\ G\Gamma_{11}^2 &= -\frac{1}{2}E_v \end{cases} \qquad \begin{cases} E\Gamma_{12}^1 &= \frac{1}{2}E_v \\ G\Gamma_{12}^2 &= \frac{1}{2}G_u \end{cases} \qquad \begin{cases} E\Gamma_{22}^1 &= -\frac{1}{2}G_u \\ G\Gamma_{22}^2 &= \frac{1}{2}G_v \end{cases}$$

or in our case (E and G are functions of v only).

$$\begin{cases} \frac{1}{v^2}\Gamma_{11}^1 &= 0 \\ \frac{1}{v^2}\Gamma_{11}^2 &= \frac{1}{v^3} \end{cases} \qquad \begin{cases} \frac{1}{v^2}\Gamma_{12}^1 &= -\frac{1}{v^3} \\ \frac{1}{v^2}\Gamma_{12}^2 &= 0 \end{cases} \qquad \begin{cases} \frac{1}{v^2}\Gamma_{22}^1 &= 0 \\ \frac{1}{v^2}\Gamma_{22}^2 &= -\frac{1}{v^3} \end{cases}$$

or

$$\begin{cases} \Gamma_{11}^1 &= 0 \\ \Gamma_{11}^2 &= \frac{1}{v} \end{cases} \qquad \begin{cases} \Gamma_{12}^1 &= -\frac{1}{v} \\ \Gamma_{12}^2 &= 0 \end{cases} \qquad \begin{cases} \Gamma_{22}^1 &= 0 \\ \Gamma_{22}^2 &= -\frac{1}{v}. \end{cases}$$

Therefore,

$$\begin{aligned} \boldsymbol{x}_{uu} &= \Gamma_{11}^1\boldsymbol{x}_u + \Gamma_{11}^2\boldsymbol{x}_v + L\boldsymbol{N} = \frac{1}{v}\boldsymbol{x}_v + L\boldsymbol{N} \\ \boldsymbol{x}_{uv} &= \Gamma_{12}^1\boldsymbol{x}_u + \Gamma_{12}^2\boldsymbol{x}_v + M\boldsymbol{N} = -\frac{1}{v}\boldsymbol{x}_u + M\boldsymbol{N} \\ \boldsymbol{x}_{vv} &= \Gamma_{22}^1\boldsymbol{x}_u + \Gamma_{22}^2\boldsymbol{x}_v + N\boldsymbol{N} = -\frac{1}{v}\boldsymbol{x}_v + N\boldsymbol{N} \end{aligned}$$

Step 2 — Calculate $LN - M^2$:

$$\begin{aligned} LN - M^2 &= L\boldsymbol{N} \cdot N\boldsymbol{N} - M\boldsymbol{N} \cdot M\boldsymbol{N} \\ &= (\boldsymbol{x}_{uu} - \frac{1}{v}\boldsymbol{x}_v) \cdot (\boldsymbol{x}_{vv} + \frac{1}{v}\boldsymbol{x}_v) - (\boldsymbol{x}_{uv} + \frac{1}{v}\boldsymbol{x}_u) \cdot (\boldsymbol{x}_{uv} + \frac{1}{v}\boldsymbol{x}_u) \\ &= \boldsymbol{x}_{uu} \cdot \boldsymbol{x}_{vv} - \boldsymbol{x}_{uv} \cdot \boldsymbol{x}_{uv} \qquad - \frac{1}{v}\underbrace{\boldsymbol{x}_{vv} \cdot \boldsymbol{x}_v}_{=G_v/2=-1/v^3} + \frac{1}{v}\underbrace{\boldsymbol{x}_{uu} \cdot \boldsymbol{x}_v}_{=F_u - E_v/2 = 1/v^3} - \frac{1}{v^2}\underbrace{\boldsymbol{x}_v \cdot \boldsymbol{x}_v}_{=G=1/v^2} \\ &\qquad - 2\frac{1}{v}\underbrace{\boldsymbol{x}_{uv} \cdot \boldsymbol{x}_u}_{=E_v/2=-1/v^3} - \frac{1}{v^2}\underbrace{\boldsymbol{x}_u \cdot \boldsymbol{x}_u}_{=E=1/v^2} \\ &= \boldsymbol{x}_{uu} \cdot \boldsymbol{x}_{vv} - \boldsymbol{x}_{uv} \cdot \boldsymbol{x}_{uv} \qquad + \frac{2}{v^4}. \end{aligned}$$

We now have

$$\begin{aligned} \boldsymbol{x}_{uu} \cdot \boldsymbol{x}_{vv} - \boldsymbol{x}_{uv} \cdot \boldsymbol{x}_{uv} &= (\boldsymbol{x}_u \cdot \boldsymbol{x}_{vv})_u - (\boldsymbol{x}_u \cdot \boldsymbol{x}_{uv})_v \\ &= (F_v - \frac{1}{2}G_u)_u - \frac{1}{2}E_{vv} = -\frac{\partial^2}{\partial v^2}\frac{1}{2v^2} = -\frac{3}{v^4}. \end{aligned}$$

Step 3 — Calculate K: Since $EG - F^2 = 1/v^4$, we have finally

$$K = \frac{LN - M^2}{EG - F^2} = \frac{-3/v^4 + 2/v^4}{1/v^4} = -1.$$

As a result, we have: the hyperbolic plane has constant curvature -1.

Remark 10.8.

(a) In Example 10.7 (or more generally, in all examples where we calculate the Gauss curvature from E, F and G only) we used the fact that $S \subset \mathbb{R}^3$ (at least locally), because we used the formulae for $\boldsymbol{x}_{uu}$ etc. involving the normal vector $\boldsymbol{N}$. This is for convenience only, to remember the procedure. More precisely, we should use the formula

$$K = \Big(\frac{LN - M^2}{EG - F^2} = \Big) \frac{E_{vv}/2 + F_{uv} - E_{vv}/2 + \text{terms in } E,F,G \text{ and derivatives}}{EG - F^2}$$

as *the definition* of K for a general surface as we did in Theorem 10.1.

(b) Recall that for plane curves the signed curvature defined a curve up to an isometry of the plane. What about a similar result for surfaces? Does the Gauss curvature define a surface uniquely (or up to what data the surface is unique)?

The answer to the uniqueness is *negative*, as Remark 10.11 shows: there exist surfaces S, $\widetilde{S}$ and a diffeomorphism $f\colon S \longrightarrow \widetilde{S}$ (f is bijective, smooth and f^{-1} is also smooth) which is *not* an isometry, but for which the Gauss curvature is preserved (i.e., $K(p) = \widetilde{K}(f(p))$, if K resp. $\widetilde{K}$ is the Gauss curvature of S resp. $\widetilde{S}$).

Example 10.9. (Gauss curvature in an orthogonal parametrization).
In an orthogonal parametrization ($F = 0$) we have

$$K = -\frac{1}{2\sqrt{EG}}\Big(\Big(\frac{E_v}{\sqrt{EG}}\Big)_v + \Big(\frac{G_u}{\sqrt{EG}}\Big)_u\Big)$$

Example 10.10. (Flat torus in $\mathbb{R}^4$).
Let $T = S^1 \times S^1 \subset \mathbb{R}^4$ be the so-called *flat torus*. We have a standard parametrization

$$\boldsymbol{x}(u,v) = (\cos u, \sin u, \cos v, \sin v), \qquad (u,v) \in U$$

with $U = (0, 2\pi) \times (0, 2\pi)$ (and other suitable sets to cover all of S).
We have

$$\boldsymbol{x}_u = (-\sin u, \cos u, 0, 0) \qquad \text{and} \qquad \boldsymbol{x}_v = (0, 0, -\sin v, \cos v),$$

so that $E = G = 1$ and $F = 0$.
Therefore the Gauss curvature is

$$K = -\frac{1}{2\sqrt{EG}}\Big(\Big(\frac{E_v}{\sqrt{EG}}\Big)_v + \Big(\frac{G_u}{\sqrt{EG}}\Big)_u\Big) = 0.$$

Example 10.11. (Surfaces with the same Gauss curvature are not necessarily isometric).
Let $U = (0, 2\pi) \times (0, \infty)$ and let S, $\widetilde{S}$ be the surfaces defined by $S = \boldsymbol{x}(U)$, $\widetilde{S} = \boldsymbol{y}(U)$, where $\boldsymbol{x}, \boldsymbol{y}\colon U \longrightarrow \mathbb{R}^3$ are defined by

$$\boldsymbol{x}(u,v) = (v\cos u, v\sin u, u), \quad \boldsymbol{y}(u,v) = (v\cos u, v\sin u, \log v), \quad (u,v) \in U.$$

(thus S is an open subset of the helicoid and $\widetilde{S}$ is an open subset of a surface of revolution).
The coefficients of the first fundamental forms of S resp. $\widetilde{S}$ w.r.t. $\boldsymbol{x}$ resp. $\boldsymbol{y}$ are

$$E = v^2 + 1, \quad F = 0, \quad G = 1 \qquad \text{and} \qquad \widetilde{E} = v^2, \quad \widetilde{F} = 0, \quad \widetilde{G} = 1 + \frac{1}{v^2}.$$

Calculating the Gauss curvature for S and $\widetilde{S}$ gives

$$K(\boldsymbol{x}(u,v)) = \widetilde{K}(\boldsymbol{y}(u,v)) = -\frac{1}{(v^2+1)^2},$$

and hence $K(p) = \widetilde{K}(f(p))$.

Since the coefficients of the first fundamental form S and $\widetilde{S}$ are different, f cannot be a local isometry (note that $f \circ \boldsymbol{x} = \boldsymbol{y}$, so that $(f \circ \boldsymbol{x})_u \cdot (f \circ \boldsymbol{x})_u = \boldsymbol{y}_u \cdot \boldsymbol{y}_u = \widetilde{E}$ etc.), so since $E \neq \widetilde{E}$, f cannot be an isometry by Proposition 8.15.