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Differential Geometry III, Term 2 (Section 11)

11 Curves on surfaces

11.1 Coordinate curves

Definition 11.1. Let S be a regular surface in $\mathbb{R}^n$. A *curve on the surface* S is a smooth map $\boldsymbol{\alpha}\colon I \longrightarrow S$ ($I \subset \mathbb{R}$ is an interval).

Remark 11.2. Recall: If $\boldsymbol{x}\colon U \longrightarrow S$ is a local parametrisation ($U \subset \mathbb{R}^2$ open) and $\boldsymbol{\alpha}\colon I \longrightarrow \boldsymbol{x}(U)$ a curve in $\boldsymbol{x}(U) \subset U$, then we can write

$$\boldsymbol{\alpha}(s) = \boldsymbol{x}(u(s), v(s)),$$

and

$$\boldsymbol{\alpha}' = u'\boldsymbol{x}_u + v'\boldsymbol{x}_v,$$

which implies

$$\|\boldsymbol{\alpha}'(t)\| = \sqrt{E(u(t), v(t))u'(t)^2 + 2F(u(t), v(t))u'(t)v'(t) + \dots}$$

Example 11.3. Coordinate curves: Let $\boldsymbol{x}\colon U \longrightarrow S$ be a local parametrization ($U \subset \mathbb{R}^2$ open) and $(u_0, v_0) \in U$, then

$$u \mapsto \boldsymbol{x}(u, v_0)$$
$$v \mapsto \boldsymbol{x}(u_0, v)$$

are called *coordinate curves* through $p = \boldsymbol{x}(u_0, v_0)$. The local parametrization is given by $(u(s), v(s)) = (s, v_0)$ for the first, and $(u(s), v(s)) = (u_0, s)$ for the second.

One should note that coordinate curves are not intrinsic, they depend on the parametrization.

11.2 Geodesic and normal curvature

Assume now that $S \subset \mathbb{R}^3$, $\boldsymbol{\alpha}\colon I \longrightarrow S \subset \mathbb{R}^3$ is a unit speed curve. Then $\boldsymbol{\alpha}'(s)$ and $\boldsymbol{\alpha}''(s)$ are orthogonal, and

$$\|\boldsymbol{\alpha}''(s)\| = \kappa(s),$$

where $\kappa(s)$ denotes the *curvature* of $\boldsymbol{\alpha}$ as a space curve.

Denote by $\boldsymbol{N}(\boldsymbol{\alpha}(s))$ the Gauss map of the surface S at $\boldsymbol{\alpha}(s)$. Since $\boldsymbol{\alpha}''$ is orthonormal to $\boldsymbol{\alpha}'$, it lies in the plane spanned by $\boldsymbol{N}$ and $\boldsymbol{N} \times \boldsymbol{\alpha}'$.

Definition 11.4 (Geodesic and normal curvature)**.** If $\boldsymbol{\alpha}\colon I \longrightarrow S$ is a curve on a surface S (with Gauss map $\boldsymbol{N}$) parametrized by arc lenth, then we can write

$$\boldsymbol{\alpha}''(s) = \kappa_{\mathrm{g}}(s)\boldsymbol{N}(\boldsymbol{\alpha}(s)) \times \boldsymbol{\alpha}'(s) + \kappa_{\mathrm{n}}(s)\boldsymbol{N}(\boldsymbol{\alpha}(s)).$$

We call $\kappa_{\mathrm{g}}\colon I \longrightarrow \mathbb{R}$ the *geodesic curvature* and $\kappa_{\mathrm{n}}\colon I \longrightarrow \mathbb{R}$ the *normal curvature* of $\boldsymbol{\alpha}$ in S.

For a curve with an arbitrary parametrization on S the geodesic and normal curvatures are defined to be the same as for its unit speed reparametrization, i.e. if $\boldsymbol{\beta} : J \to S$ is a curve, $\boldsymbol{\alpha} : I \to S$ is a unit speed curve, and $\boldsymbol{\beta}(t(s)) = \boldsymbol{\alpha}(s)$, then $\kappa_{\boldsymbol{\beta},\mathrm{n}}(t(s)) = \kappa_{\boldsymbol{\alpha},\mathrm{n}}(s)$, and $\kappa_{\boldsymbol{\beta},\mathrm{g}}(t(s)) = \kappa_{\boldsymbol{\alpha},\mathrm{g}}(s)$. In other words, normal and geodesic curvatures are invariant under reparametrizations by definition.

Remark 11.5. We have (if $\boldsymbol{\alpha}$ is parametrized by arc length!)

$$\kappa_{\mathrm{n}} = \boldsymbol{\alpha}'' \cdot \boldsymbol{N} \qquad \text{and} \qquad \kappa_{\mathrm{g}} = \boldsymbol{\alpha}'' \cdot (\boldsymbol{N} \times \boldsymbol{\alpha}')$$

Furthermore, recall that the curvature κ of a *space curve* is given by $\kappa = \|\boldsymbol{\alpha}''\|$ (if $\boldsymbol{\alpha}$ is parametrized by arc length), and since $\boldsymbol{N}$ and $\boldsymbol{N} \times \boldsymbol{\alpha}'$ form an orthonormal system, we have by Pythagoras' Theorem

$$\kappa = \|\boldsymbol{\alpha}''\| = \sqrt{\kappa_{\mathrm{g}}^2 + \kappa_{\mathrm{n}}^2}$$

Example 11.6. (a) (Plane).

$S = \{\, (u, v, 0) \mid (u, v) \in \mathbb{R}^2 \,\}$, then $\boldsymbol{N} = (0, 0, 1)$.

Let $\boldsymbol{\alpha} \colon I \longrightarrow S$, $\boldsymbol{\alpha}(s) = (u(s), v(s), 0)$, parametrized by arclength; then $\boldsymbol{\alpha}' = (u', v', 0)$, $\boldsymbol{n} \times \boldsymbol{\alpha}' = (-v', u', 0)$ so that

$$\boldsymbol{\alpha}'' = (u'', v'', 0) = \kappa_{\mathrm{g}}(\boldsymbol{N} \times \boldsymbol{\alpha}') + \kappa_{\mathrm{n}} \boldsymbol{N} = \kappa_{\mathrm{g}}(-v', u', 0) + \kappa_{\mathrm{n}}(0, 0, 1)$$

so that $\kappa_{\mathrm{n}} = 0$, and, if κ is the curvature of $\boldsymbol{\alpha}$, $\kappa = \kappa_{\mathrm{g}}$ (if $\boldsymbol{\alpha}$ is considered as a plane curve) or $\kappa = |\kappa_{\mathrm{g}}|$ (if $\boldsymbol{\alpha}$ is considered as a space curve).

(b) (Lines on surfaces).

Assume that $\boldsymbol{\alpha}(s) = p + s\boldsymbol{v}$, $\|v\| = 1$, parametrizes a line ($s \in I \subset \mathbb{R}$) and that $\boldsymbol{\alpha}(s) \in S$ for all $s \in I$ for some surface $S \subset \mathbb{R}^3$. Then

$$\boldsymbol{\alpha}' = \boldsymbol{v}, \qquad \boldsymbol{\alpha}'' = (0, 0, 0),$$

so that $\kappa_{\mathrm{g}} = 0$ and $\kappa_{\mathrm{n}} = 0$, i.e., the geodesic and normal curvature of a line on a surface both vanish.

Theorem 11.7 (Meusnier)**.** All curves $\boldsymbol{\beta}$ through $p \in S$ with the same tangent vector $\boldsymbol{w} \in T_pS$ have the same normal curvature

$$\kappa_{\mathrm{n}}(s) = I\!I_p\Big(\frac{\boldsymbol{w}}{\|\boldsymbol{w}\|}\Big).$$

In particular, the value $\kappa_{\mathrm{n}}(\boldsymbol{w})$ is well defined for any $\boldsymbol{w} \in T_pS$.

Corollary. Let $p \in S$, $\boldsymbol{w} \in T_pS$, and let Π be the plane through p spanned by $\boldsymbol{N}(p)$ and $\boldsymbol{w}$. Then $\kappa_{\mathrm{n}}(\boldsymbol{w}) = \kappa(\Pi \cap S)$, where $\Pi \cap S$ is considered as a plane curve with tangent vector $\boldsymbol{w}$ at p.

Proposition 11.8. (Normal curvature in a local parametrization)

Let S be a surface in $\mathbb{R}^3$, and let E, F, G and L, M, N be the coefficient of the first and second fundamental forms respectively w.r.t. a parametrization $\boldsymbol{x}$. Further, let $\boldsymbol{\alpha}$ be a curve in S (not necessarily parametrized by arc length) with local parametrization $\boldsymbol{\alpha}(s) = \boldsymbol{x}(u(s), v(s))$. Then

$$\kappa_{\mathrm{n}} = I\!I_p\left(\frac{\boldsymbol{\alpha}'}{\|\boldsymbol{\alpha}'\|}\right) = \frac{(u')^2 L + 2u'v'M + (v')^2 N}{(u')^2 E + 2u'v'F + (v')^2 G} = \frac{I\!I_p(\boldsymbol{\alpha}')}{I_p(\boldsymbol{\alpha}')}$$

Proposition 11.9. Let $\boldsymbol{\beta} \colon I \longrightarrow S$ be a curve not necessarily parametrized by arc length, and let $\boldsymbol{N}$ be the Gauss map of S. Then the geodesic curvature of $\boldsymbol{\beta}$ can be calculated as

$$\kappa_{\mathrm{g}} = \frac{1}{\|\boldsymbol{\beta}'\|^3}(\boldsymbol{\beta}' \times \boldsymbol{\beta}'') \cdot \boldsymbol{N}.$$

Definition 11.10. (Asymptotic curves) A curve $\boldsymbol{\alpha}$ on a surface $S \subset \mathbb{R}^3$ is called an *asymptotic curve* if its normal curvature vanishes identically (i.e., if $\kappa_{\mathrm{n}} = 0$).

Remark 11.11. (i) The following are equivalent (TFAE):

(a) $\boldsymbol{\alpha}$ is an asymptotic curve;

(b) $\boldsymbol{\alpha}'' \cdot (\boldsymbol{N} \circ \boldsymbol{\alpha}) = 0$ (if $\boldsymbol{N}$ is the Gauss map of S and $\boldsymbol{\alpha}$ is parametrized by arc length);

(c) $\kappa_{\mathrm{n}} = 0$;

(d) $I\!I_{\boldsymbol{\alpha}(s)}(\boldsymbol{\alpha}'(s)) = 0$ for all s ($\boldsymbol{\alpha}$ not necessarily parametrized by arc length);

(e) $(u')^2 L + 2u'v'M + (v')^2 N = 0$ in a local parametrization $s \mapsto \boldsymbol{x}(u(s), v(s))$ of $\boldsymbol{\alpha}$.

In particular, $I\!I_p$ is not positive or negative definite along $\boldsymbol{\alpha}$, so $\boldsymbol{\alpha}$ has to be in the *hyperbolic* or *flat* region of the surface.

(ii) $\kappa_{\mathrm{n}}(\boldsymbol{w}) = 0$ for $\boldsymbol{w} \in T_pS$ implies $K(p) \le 0$.

(iii) If $\boldsymbol{\alpha}$ is a line on S, then $\kappa_{\mathrm{n}} = 0$, i.e., any line on a surface is an asymptotic curve.

Example 11.12. (Asymptotic curves on a surface of revolution/catenoid)

Recall that on a surface of revolution obtained by rotating a curve $\boldsymbol{\alpha}$ given by $\boldsymbol{\alpha}(v) = (f(v), 0, g(v))$ around the z-axis, we have

$$L = \frac{-fg'}{\|\boldsymbol{\alpha}'\|}, \quad M = 0, \quad N = \frac{f''g' - f'g''}{\|\boldsymbol{\alpha}'\|}$$

(see Example 9.13). A curve $\boldsymbol{\beta}$ parametrized locally by $\boldsymbol{\beta}(t) = \boldsymbol{x}(u(t), v(t))$ is an asymptotic curve iff $(u')^2 L + 2u'v'M + (v')^2 N = 0$, i.e., iff

$$(u')^2 fg' = (v')^2 (f''g' - f'g'')$$

If in particular, $f(v) = \cosh v$ and $g(v) = v$ (i.e., the surface of revolution is a *catenoid*), then the above equation becomes

$$(u')^2 \cosh v = (v')^2 \cosh v, \quad \text{or,} \quad u' = \pm v', \quad \text{i.e.,} \quad u = \pm v + c$$

for some constant $c \in \mathbb{R}$.

11.3 Lines of curvature

Definition 11.13. (Lines of curvature)

A curve $\boldsymbol{\alpha}\colon I \longrightarrow S$ on a surface S in $\mathbb{R}^3$ is called a *line of curvature* if $\boldsymbol{\alpha}'(s)$ is a principal direction at $\boldsymbol{\alpha}(s)$ for all $s \in I$, i.e., $\boldsymbol{\alpha}'(s)$ is an eigenvector of the Weingarten map at $\boldsymbol{\alpha}(s)$ for all s.

Equivalently, $\boldsymbol{\alpha}$ is a line of curvature if there is a function $\lambda\colon I \longrightarrow \mathbb{R}$ such that

$$-d\boldsymbol{N}_{\boldsymbol{\alpha}(s)}(\boldsymbol{\alpha}'(s)) = \lambda(s)\boldsymbol{\alpha}'(s)$$

for all $s \in I$. (Here $\lambda(s)$ is a principal curvature at $\boldsymbol{\alpha}(s)$.)

Remark 11.14. Note that if the eigenvalues of a symmetric 2×2-matrix are different, then the corresponding eigenvectors are orthogonal. Hence, each non-umbilic point ($\kappa_1(p) \neq \kappa_2(p)$) has two lines of curvature through it, and they intersect orthogonally. In an umbilic point, this family of orthogonally intersecting curves has a singularity.

Moreover any direction at an umbilic point is principal. In particular, on a sphere ($\kappa_1 = \kappa_2 > 0$) or a plane ($\kappa_1 = \kappa_2 = 0$) any curve is a line of curvature.

Proposition 11.15. (Lines of curvature in a local parametrisation) Let E, F, G and L, M, N be the coefficients of the first and second fundamental forms respectively w.r.t. a local parametrization $\boldsymbol{x}: U \longrightarrow S$, and let $\boldsymbol{\alpha}$ be a curve in S with local parametrization $\boldsymbol{\alpha}(s) = \boldsymbol{x}(u(s), v(s))$. Then $\boldsymbol{\alpha}$ is a line of curvature if and only if

$$\det \begin{pmatrix} (v')^2 & -u'v' & (u')^2 \\ E & F & G \\ L & M & N \end{pmatrix} = 0$$

or, equivalently,

$$(FN - GM)(v')^2 + (EN - GL)u'v' + (EM - FL)(u')^2 = 0.$$

Example 11.16. (Hyperbolic paraboloid)

Let $S = \{\, (x, y, z) \mid xy = z \,\}$ be a hyperbolic paraboloid parametrized by $\boldsymbol{x}(u, v) = (u, v, uv)$. Then

$$\boldsymbol{x}_u = (1, 0, v), \quad \boldsymbol{x}_v = (0, 1, u), \quad \boldsymbol{N} = D^{-1}(-v, -u, 1), \quad D = (u^2 + v^2 + 1)^{1/2}$$
$$\boldsymbol{x}_{uu} = (0, 0, 0), \quad \boldsymbol{x}_{uv} = (0, 0, 1), \quad \boldsymbol{x}_{vv} = (0, 0, 0)$$

and

$$E = \boldsymbol{x}_u \cdot \boldsymbol{x}_u = 1 + v^2, \quad F = \boldsymbol{x}_u \cdot \boldsymbol{x}_v = uv, \quad G = \boldsymbol{x}_v \cdot \boldsymbol{x}_v = 1 + u^2,$$
$$L = \boldsymbol{x}_{uu} \cdot \boldsymbol{N} = 0, \quad M = \boldsymbol{x}_{uv} \cdot \boldsymbol{N} = 1/D, \quad N = \boldsymbol{x}_{vv} \cdot \boldsymbol{N} = 0$$

Therefore, $\boldsymbol{\alpha}$ with $\boldsymbol{\alpha}(s) = \boldsymbol{x}(u(s), v(s))$ is a *line of curvature* iff

$$\det \begin{pmatrix} (v')^2 & -u'v' & (u')^2 \\ 1 + v^2 & uv & 1 + u^2 \\ 0 & 1/D & 0 \end{pmatrix} = (u')^2(1 + v^2)/D - (v')^2(1 + u^2)/D = 0,$$

which is equivalent to

$$\frac{u'}{(1 + u^2)^{1/2}} = \pm \frac{v'}{(1 + v^2)^{1/2}},$$

and after integrating,

$$\operatorname{arcsinh} u = \pm \operatorname{arcsinh} v + c$$

for some constant $c \in \mathbb{R}$. For example, if $c = 0$, then $u = \pm v$, or $s \mapsto \boldsymbol{x}(s, \pm s) = (s, \pm s, \pm s^2)$ are the lines of curvature through $p = (0, 0, 0)$.

The *asymptotic curves* here are given by

$$(u')^2 L + 2u'v'M + (v')^2 M = 2u'v'/D = 0,$$

i.e., $u' = 0$ or $v' = 0$, so the asymptotic curves are the coordinate curves $s \mapsto \boldsymbol{x}(s, v_0)$ or $s \mapsto \boldsymbol{x}(u_0, s)$

Remark 11.17. (a) On a *line of curvature*, the *normal curvature* is a *principal curvature.*

Indeed, since $\boldsymbol{\alpha}$ is a line of curvature, we have $-d_{\boldsymbol{\alpha}(s)}\boldsymbol{N}(\boldsymbol{\alpha}'(s)) = \lambda(s)\boldsymbol{\alpha}'(s)$, and $\lambda(s)$ is a principal curvature at $\boldsymbol{\alpha}(s)$.

On the other hand,

$$\kappa_{\mathrm{n}}(s) = \frac{I\!I_{\boldsymbol{\alpha}(s)}(\boldsymbol{\alpha}'(s))}{I_{\boldsymbol{\alpha}(s)}(\boldsymbol{\alpha}'(s)} = \frac{\langle -d_{\boldsymbol{\alpha}(s)}\boldsymbol{N}(\boldsymbol{\alpha}'(s)), \boldsymbol{\alpha}'(s)\rangle}{\langle \boldsymbol{\alpha}'(s), \boldsymbol{\alpha}'(s)\rangle} = \frac{\langle \lambda(s)\boldsymbol{\alpha}'(s), \boldsymbol{\alpha}'(s)\rangle}{\langle \boldsymbol{\alpha}'(s), \boldsymbol{\alpha}'(s)\rangle} = \lambda(s)$$

(b) Assume that a line $\boldsymbol{\alpha}$ (or a part of it) belongs to a surface. When is this line a *line of curvature*?

On a line, the normal curvature is 0, hence by the first part, one of its principal curvatures, say κ_1, has to vanish on $\boldsymbol{\alpha}$. But this means that the Gauss curvature (as the product of the two principal curvatures $K = \kappa_1\kappa_2$) has to vanish (and vice versa). Hence if $\boldsymbol{\alpha}\colon I \longrightarrow S$ is a line in S, then

$$\boldsymbol{\alpha} \text{ is a line of curvature} \quad \Leftrightarrow \quad \big(K(\boldsymbol{\alpha}(s)) = 0 \quad \forall s \in I\big).$$

This is equivalent to $LN - M^2 = 0$.

Proposition 11.18. (Lines of curvature for a principal parametrization)
If $\boldsymbol{x}$ is a principal parametrization of a surface $S \subset \mathbb{R}^3$ (i.e., $F = 0$ and $M = 0$), then the coordinate curves are lines of curvature.

Example 11.19. (Lines of curvature for a surface of revolution)
On a surface of revolution, the coordinate curves of the standard parametrization given by $\boldsymbol{x}(u, v) = (f(v)\cos u, f(v)\sin u, g(v))$ are also lines of curvature.

Remark 11.20. Note that the converse of Proposition 11.18 is also true in the following sense: if a parametrization $\boldsymbol{x}$ is principal and the umbilic points are isolated, then the lines of curvature are coordinate curves.