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Differential Geometry III, Term 2 (Section 12)

12 Geodesics

Definition 12.1. Let $\boldsymbol{\alpha}\colon I \longrightarrow S$ be a (regular) curve on a surface $S \subset \mathbb{R}^3$. $\boldsymbol{\alpha}$ is called *geodesic* if $\boldsymbol{\alpha}''$ is normal to S (i.e., $\boldsymbol{\alpha}''(s)$ is orthogonal to $T_{\boldsymbol{\alpha}(s)}S$ for all $s \in I$).

Note that the curve does not need to be parametrized by arc length, but we have:

Proposition 12.2 (Geodesics have constant speed)**.** Let $\boldsymbol{\alpha}$ be a geodesic, then $\|\boldsymbol{\alpha}'\|$ is constant, i.e., there exists $c > 0$ such that $\boldsymbol{\alpha}'(s) = c$ for all $s \in I$.

In other words, a geodesic is always parametrized *proportionally* to arc length.

Example 12.3.

(a) **Lines are geodesics.**

Let S be a surface and $\boldsymbol{\alpha}$ be a line in S. Then $\boldsymbol{\alpha}''(s) = 0$, hence $\boldsymbol{\alpha}''$ is normal to any vector (in particular to the tangent plane $T_{\boldsymbol{\alpha}(s)}S$). Therefore, $\boldsymbol{\alpha}$ is a geodesic.

(b) **Geodesics on a cylinder.**

Let $S = \{\, (x, y, z) \,|\, x^2 + y^2 = 1 \,\}$, then any geodesic $\boldsymbol{\alpha}$ on S is parametrized by

$$\boldsymbol{\alpha}(s) = (\cos(as + b), \sin(as + b), \lambda s + \mu)$$

for some $\lambda, \mu, a, b \in \mathbb{R}$. If $a = 0$ then $\boldsymbol{\alpha}$ is a meridian, if $\lambda = 0$ then $\boldsymbol{\alpha}$ is a parallel.

(c) **Great circles on a sphere are geodesics.**

A *great circle* on a sphere is the curve given by the intersection of the sphere with a plane through its origin.

Let $S = \{\, (x, y, z) \,|\, x^2 + y^2 + z^2 = 1 \,\}$, and let $\boldsymbol{v}, \boldsymbol{w}$ be orthonormal in $\mathbb{R}^3$. Set

$$\boldsymbol{\alpha}(s) = (\cos s)\boldsymbol{v} + (\sin s)\boldsymbol{w}$$

for $s \in I$ (I some interval). Then $\boldsymbol{\alpha}''(s) = -\boldsymbol{\alpha}(s) = -\boldsymbol{N}(\boldsymbol{\alpha}(s))$, hence $\boldsymbol{\alpha}$ is orthogonal to $T_{\boldsymbol{\alpha}(s)}S$ and $\boldsymbol{\alpha}$ is a geodesic.

Proposition 12.4 (Equivalent characterization of geodesics)**.** The following are equivalent (TFAE):

(a) $\boldsymbol{\alpha}$ is a geodesic;

(b) $\boldsymbol{\alpha}$ has constant speed and its geodesic curvature vanishes.

Proposition 12.5 (Geodesics in a local parametrization)**.** Let $\boldsymbol{\alpha}\colon I \longrightarrow S$ be a curve on a surface $S \subset \mathbb{R}^3$, and let $\boldsymbol{x}\colon U \longrightarrow S$ be a local parametrization. We write $\boldsymbol{\alpha}(s) = \boldsymbol{x}(u(s), v(s))$ and E, F, G for the coefficients of the first fundamental form w.r.t. $\boldsymbol{x}$. Then the following are equivalent:

(a) $\boldsymbol{\alpha}$ is a geodesic;

(b) $\boldsymbol{\alpha}'' \cdot \boldsymbol{x}_u = 0$ and $\boldsymbol{\alpha}'' \cdot \boldsymbol{x}_v = 0$;

(c)

$$u''E + \frac{1}{2}(u')^2 E_u + u'v'E_v + (v')^2\Big(F_v - \frac{1}{2}G_u\Big) + v''F = 0,$$
$$v''G + \frac{1}{2}(v')^2 G_v + u'v'G_u + (u')^2\Big(F_u - \frac{1}{2}E_v\Big) + u''F = 0.$$

Let us now state the main theorem about geodesics:

Theorem 12.6 (Local existence and uniqueness of geodesics)**.** (a) Let $p \in S$, $\boldsymbol{w} \in T_pS \setminus \{0\}$. Then there exists $\varepsilon > 0$ and a *unique* geodesic $\boldsymbol{\alpha}\colon (-\varepsilon, \varepsilon) \longrightarrow S$ such that $\boldsymbol{\alpha}(0) = p$ and $\boldsymbol{\alpha}'(0) = \boldsymbol{w}$.

(b) Geodesics are determined entirely by the coefficients of the first fundamental form E, F and G (and their derivatives) in a local parametrization.

Corollary 12.7 (Isometries take geodesics to geodesics)**.** Let $f\colon S \longrightarrow \widetilde{S}$ be a local isometry between two surfaces S and $\widetilde{S}$, and let $\boldsymbol{\alpha}\colon I \longrightarrow S$ be a geodesic on S. Then $f \circ \boldsymbol{\alpha}\colon I \longrightarrow \widetilde{S}$ is also a geodesic on $\widetilde{S}$.

Example 12.8.

(a) **Plane.**

We know that $E = G = 1$ and $F = 0$ (in the standard parametrization $(u, v) \in \mathbb{R}^2$), so the local equation for a geodesic is

$$u'' = 0 \qquad \text{and} \qquad v'' = 0$$

This means that

$$u(s) = u_0 + as \qquad \text{and} \qquad v(s) = v_0 + bs$$

for some numbers u_0, v_0, a, b ((u_0, v_0) is the starting point and $\boldsymbol{w} = (a, b)$ is the initial speed vector). These are all geodesics on a plane

(b) **Cylinder.**

Let $S := \{\,(x, y, z) \mid x^2 + y^2 = 1\,\}$ be a cylinder and $f\colon \mathbb{R}^2 \longrightarrow S$ be given by $f(u, v) = (\cos u, \sin u, v)$, then f is a local isometry. Geodesics on the cylinder S are just images of lines under f:

- lines $s \mapsto (\cos u_0, \sin u_0, s)$ (u_0 some constant): image of the line $s \mapsto (u_0, s)$;
- circles $s \mapsto (\cos s, \sin s, v_0)$ (v_0 some constant): image of the line $s \mapsto (s, v_0)$;
- helices $s \mapsto (\cos s, \sin s, v_0 + as)$ (v_0, a some constants): image of the line $s \mapsto (s, v_0 + as)$ (the circles above are the case $a = 0$)

These are all geodesics (use the local *uniqueness* result of Theorem 12.6), cf. Example 12.3.

Remark 12.9 (Minimizing property of geodesics)**.** (a) The shortest curve between two points on a surface is a geodesic (if parametrized proportionally to arc length).

(b) Converse is false: not all geodesics connecting two points minimize the distance.

(c) A minimizing curve (a geodesic) might not be unique. Moreover, there might be infinitely many of these.

(d) There might be no geodesic joining two points on a surface.

Example 12.10 (Geodesics on a surface of revolution)**.** Let S be a surface of revolution with local parametrization

$$\boldsymbol{x}(u,v) = (f(v)\cos u, f(v)\sin u, g(v)),$$

and let $\boldsymbol{\alpha}(s) = \boldsymbol{x}(u(s), v(s))$ be a geodesic on S. Then the equations from Prop. 12.5 reduce to

$$u''E + u'v'E_v = 0,$$
$$v''G + \frac{1}{2}v'^2 G_v - \frac{1}{2}u'^2 E_v = 0.$$

The first equation is equivalent to $(u'E)' = 0$, or

$$u' = \frac{c}{f^2}$$

for some constant $c \in \mathbb{R}$.

Assuming that the the generating curve $(f, 0, g)$ is unit speed, the second equation is reduced to $v''G - u'^2 E_v/2 = 0$, or, equivalently,

$$v'' - u'^2 f f' = 0$$

as $E = f^2$.

Corollary. (a) All meridians are geodesics

(b) A parallel $v = v_0$ is geodesic if and only if $f'(v_0) = 0$.

Proposition 12.11 (Clairaut relation)**.** Let S be a surface of revolution with local parametrization

$$\boldsymbol{x}(u,v) = (f(v)\cos u, f(v)\sin u, g(v)),$$

and let $\boldsymbol{\alpha}(s) = \boldsymbol{x}(u(s), v(s))$ be a geodesic on S. Denote by $\Theta(s)$ the angle formed formed by $\boldsymbol{\alpha}'(s)$ and the parallel through $\boldsymbol{\alpha}(s)$. Then

$$f(v(s))\cos\Theta(s) = \text{const}$$

Example 12.12 (Torus of revolution)**.** Let S be a torus of revolution with local parametrization

$$\boldsymbol{x}(u,v) = ((R + r\cos v)\cos u, (R + r\cos v)\sin u, r\sin v)$$

for $0 < r < R$. Let $\boldsymbol{\alpha}(s)$ be a geodesic on S through a point $\boldsymbol{\alpha}(0) = (R + r, 0, 0)$. Denote by Θ_0 the angle formed by $\boldsymbol{\alpha}'(0)$ and $\boldsymbol{x}_u$. Then $\boldsymbol{\alpha}(s)$ satisfies the equation

$$(R + r\cos v(s))\cos\Theta(s) = (R + r)\cos\Theta_0$$

Definition 12.13. A geodesic $\boldsymbol{\alpha}\colon I \longrightarrow S$ is *closed* if there is $c \in \mathbb{R}_+$ such that $\boldsymbol{\alpha}(s + c) = \boldsymbol{\alpha}(s)$ for every $s \in I$.

Example 12.14. (a) Every geodesic on a sphere is closed.

(b) The only closed geodesics on a cylinder are parallels.

Example 12.15. There are no closed geodesics on an elliptic paraboloid of revolution.