Durham University
Pavel Tumarkin Epiphany 2017

Differential Geometry III, Term 2 (Section 13)

13 Gauss–Bonnet theorems

13.1 A bit of topology

Definition 13.1. (a) A surface $S \subset \mathbb{R}^n$ is a *closed surface* if S is bounded, connected and closed (as a set).

(b) A surface is *oriented* if the Gauss map can be defined globally as a continuous map.

(c) A *region* of a surface S is a subset of S such that its boundary consists of a finite number of smooth curves (called *edges*) and its interior is non-empty. We call the points in which two smooth curves meet on the boundary *vertices* (and we assume for simplicity that the curves meet non-tangentially).

(d) A *triangle* is a region with three edges and three vertices homeomorphic to a disc (note that the edges, as well as the vertices, may coincide).

(e) A *triangulation* of a (bounded) region R is a subdivision of S into a finite number of triangles meeting only in common edges or common vertices.

(f) The *Euler characteristic* of a region R is defined by

$$\begin{aligned}\chi(R) :&= F(R) - E(R) + V(R) \\ &= \#\text{triangles} - \#\text{edges} + \#\text{vertices},\end{aligned}$$

where $F(R)$ is the number of triangles, $E(R)$ the number of edges and $V(R)$ the number of vertices of the triangulation.

Example 13.2. A closed disc has Euler characteristic 1, a sphere has Euler characteristic 2, a closed cylinder $S^1 \times [0,1]$ (as well as a torus) has Euler characterisic 0.

A priori, the Euler characteristic may depend on the triangulation.

Theorem 13.3. The Euler characteristic is independent of the triangulation.

Basically, oriented closed surfaces can be topologically characterized by their Euler characteristic:

$$\chi(S) = 2 - 2g,$$

where g is the *genus* of S (roughly, the number of "handles" in S).

Theorem 13.4 (Jordan Curve Theorem)**.** Let S be a surface homeomorphic to the plane, and let $\boldsymbol{\alpha}\colon [0,1] \longrightarrow S$ be a simple closed curve (i.e., $\boldsymbol{\alpha}(0) = \boldsymbol{\alpha}(1)$ and $\boldsymbol{\alpha}(t_1) \neq \boldsymbol{\alpha}(t_2)$ for $t_1 < t_2$ other than $t_1 = 0$, $t_2 = 1$). Then $S \setminus \boldsymbol{\alpha}(I)$ has exactly two components, and one of them is homeomorphic to a disc.

13.2 The Gauss–Bonnet theorem

Definition 13.5. Let $R \subset S$ be a region.

(a) Denote by $\mathrm{d}A$ the area measure of a surface S (locally, $\mathrm{d}A = \sqrt{EG - F^2}\,\mathrm{d}u\,\mathrm{d}v$), and we will write

$$\int_R K \,\mathrm{d}A$$

for the integral of the Gauss curvature over R (the *total* Gauss curvature of R).

(b) Denote by $\mathrm{d}s$ the length measure of a curve or the boundary of a region, we will write

$$\int_{\partial R} \kappa_{\mathrm{g}} \,\mathrm{d}s = \sum_{j=1}^{r} \int_{I_j} \kappa_{\mathrm{g},\boldsymbol{\alpha}_j}(s)\,\mathrm{d}s_j$$

for the line integral of the geodesic curvature along the boundary of a region consisting of r smooth curves $\boldsymbol{\alpha}_j$.

(c) Let us parametrize the curves along ∂R counter-clockwise, and the curves are numbered in the same direction. We define the *angle* ϑ_j at the vertex v_j (where curve $\boldsymbol{\alpha}_{j-1}$ and $\boldsymbol{\alpha}_j$ meet) as the angle between the tangent vector of $\boldsymbol{\alpha}_{j-1}$ with the tangent vector of $\boldsymbol{\alpha}_j$, i.e. ϑ_j is the exterior angle of R at v_j.

Note that all objects here are intrinsic (Gauss curvature, geodesic curvature), so we can state the Gauss–Bonnet Theorem for any surface S embedded in $\mathbb{R}^n$ (not only for $n = 3$).

Theorem 13.6 (Global Gauss–Bonnet Theorem)**.** Let R be a region in an oriented surface S. Then

$$\int_R K \,\mathrm{d}A + \int_{\partial R} \kappa_{\mathrm{g}} \,\mathrm{d}s + \sum_{j=1}^{r} \vartheta_j = 2\pi\chi(R).$$

Let us mention some special cases.

Corollary 13.7 (Special cases of the Gauss–Bonnet Theorem)**.**

(a) *(R bounded by geodesics)* If the region R is bounded piecewise by *geodesics*, then

$$\int_R K \,\mathrm{d}A + \sum_{j=1}^{r} \vartheta_j = 2\pi\chi(R).$$

(b) *(R bounded by a closed geodesic)* If γ is a simple closed geodesic and R is a region having γ as its boundary, then

$$\int_R K \,\mathrm{d}A = 2\pi\chi(R).$$

(c) *(No boundary, case $R = S$, $\partial R = \emptyset$)* If S is a closed surface, then

$$\int_S K \,\mathrm{d}A = 2\pi\chi(S).$$

Theorem 13.8 (Local Gauss–Bonnet Theorem/Gauss–Bonnet Theorem for triangles)**.** Let T be a triangle in an oriented surface S with interior angles α, β and γ. Then

$$\int_T K\,\mathrm{d}A + \int_{\partial T} \kappa_{\mathrm{g}}\,\mathrm{d}s = \alpha + \beta + \gamma - \pi.$$

Some more special cases.

Corollary 13.9. Assume that S is a surface of constant Gauss curvature K. Assume additionally, that T is a geodesic triangle in S (i.e., ∂T consists of three arcs of geodesics). Then

$$K \cdot (\operatorname{area} T) = \alpha + \beta + \gamma - \pi.$$

Example 13.10.

(a) On a sphere ($K = 1$), the sum of angles in a (geodesic) triangle is always *larger* than π and the difference is equal to the area of the triangle.

(b) On a plane ($K = 0$), the sum of angles in a (geodesic) triangle is always π (independent of the area of the triangle).

(c) On the hyperbolic plane ($K = -1$), the sum of angles in a (geodesic) triangle is always *smaller* than π and the difference is equal to the area of the triangle.

Example 13.11. (a) The total Gauss curvature of the region R of a unit sphere given by the triangle with vertices at the North pole and two points on the equator at distance one quarter of the circumference is equal to $\pi/2$ as R covers one eighth of the surface of the unit sphere. On the other hand, one can observe that R is a regular right-angled triangle, so the statement of the local Gauss–Bonnet theorem becomes "area of $R = 3\pi/2 - \pi$".

(b) The total Gauss curvature of a surface T homeomorphic to a torus is equal to zero since the Euler characteristic is zero. In particular, if T is not flat everywhere, then it contains elliptic, parabolic and flat points.

Example 13.12. Let S be homeomorphic to the plane $\mathbb{R}^2$, and assume that $K \leq 0$ everywhere on S. Then S cannot have any simple closed geodesic.

Indeed, by the Jordan curve theorem, a simple closed curve $\boldsymbol{\alpha}$ encloses two regions, one of them homeomorphic to a disc; call this region R. If we assume now that $\boldsymbol{\alpha}$ were a closed geodesic, then its geodesic curvature would vanish and there would be no vertices, hence by the Gauss–Bonnet theorem we would have

$$\int_R K\,\mathrm{d}A + \underbrace{\int_{\partial R} \kappa_{\mathrm{g}}\,\mathrm{d}s}_{=0} + \underbrace{\sum_{j=1}^{r} \vartheta_j}_{=0} = 2\pi \underbrace{\chi(R)}_{=1}$$

as the Euler characteristic of a disc is $\chi(R) = 1$ (the same as for a triangle). But since $K \leq 0$, the integral $\int_R K\,\mathrm{d}A \leq 0$, and this is a contradiction. Therefore, there is no such geodesic.

Example 13.13. One can verify the local Gauss–Bonnet theorem explicitly for an "ideal" triangle on a hyperbolic plane: the area of the region bounded by two vertical lines $u = u_1$ and $u = u_2$ and a semicircle intersecting the real axis at points u_1 and u_2 is equal to π.

Example 13.14. Let T be a flat torus in $\mathbb{R}^4$ (i.e. a torus parametrized by $\boldsymbol{x}(u, v) = (\cos u, \sin u, \cos v, \sin v)$). The Gauss–Bonnet theorem implies that any non-closed geodesic on T is not self-intersecting.

The same result can be obtained by considering the geodesics on T as images of lines on $\mathbb{R}^2$ under local isometry $\boldsymbol{x}$.