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Differential Geometry III, Term 2 (Section 9)

9 Geometry of the Gauss map

9.1 The Weingarten map

Lemma 9.1. Let S be a surface in $\mathbb{R}^3$ and $\boldsymbol{N}\colon S \longrightarrow S^2$ be its Gauss map. Then $d_p\boldsymbol{N}(\boldsymbol{w})$ is orthogonal to $\boldsymbol{N}(p)$ for every $\boldsymbol{w} \in T_pS$. In particular, we can identify $T_{\boldsymbol{N}(p)}S^2$ and T_pS, and consider $d_p\boldsymbol{N}$ as a map

$$d_p\boldsymbol{N}\colon T_pS \longrightarrow T_pS.$$

Moreover, $d_p\boldsymbol{N}$ is *symmetric*, i.e.,

$$\langle d_p\boldsymbol{N}(\boldsymbol{w}_1), \boldsymbol{w}_2\rangle = \langle \boldsymbol{w}_1, d_p\boldsymbol{N}(\boldsymbol{w}_2)\rangle$$

for all $\boldsymbol{w}_1, \boldsymbol{w}_2 \in T_pS$.

Definition 9.2.

(a) The map $-d_p\boldsymbol{N}\colon T_pS \longrightarrow T_pS$ is called the *Weingarten map* of the surface $S \subset \mathbb{R}^3$ at $p \in S$.

(b) The quadratic form $I\!I_p\colon T_pS \longrightarrow \mathbb{R}$, $I\!I_p(\boldsymbol{w}) = \langle -d_p\boldsymbol{N}(\boldsymbol{w}), \boldsymbol{w}\rangle$, is called the *second fundamental form of S at p*.

Remark 9.3. Since $-d_p\boldsymbol{N}$ is symmetric, the Weingarten map is diagonalizable in an orthogonal basis of T_pS.

Since $-d_p\boldsymbol{N}$ is now a linear operator on the tangent space T_pS, we can calculate its characteristic polynomial, trace, determinant and eigenvalues (these do not depend on a basis).

Definition 9.4. Let S be a regular surface in $\mathbb{R}^3$ with Gauss map $\boldsymbol{N}\colon S \longrightarrow S^2$ and Weingarten map $-d_p\boldsymbol{N}\colon T_pS \longrightarrow T_pS$ at $p \in S$.

(a) $K(p) = \det(-d_p\boldsymbol{N})$ is called the *Gauss curvature of S at p*.

(b) $H(p) = \frac{1}{2}\operatorname{tr}(-d_p\boldsymbol{N})$ is called the *mean curvature of S at p*.

(c) The eigenvalues $\kappa_1(p)$, $\kappa_2(p)$ of $-d_p\boldsymbol{N}$ are called *principal curvatures* of S at p.

(d) The eigenvectors $\boldsymbol{e}_1(p)$, $\boldsymbol{e}_2(p)$ of $-d_p\boldsymbol{N}$ are called *principal directions* of S at p (i.e., $-d_p\boldsymbol{N}(\boldsymbol{e}_i(p)) = \kappa_i(p)\boldsymbol{e}_i(p)$).

Remark 9.5. Obviously, we have

$$K(p) = \kappa_1(p)\kappa_2(p), \quad H(p) = \frac{1}{2}\big(\kappa_1(p) + \kappa_2(p)\big).$$

Example 9.6 (Sphere)**.** Let $S = S^2(r)$ for some $r > 0$ be a sphere. The normal vector at $\boldsymbol{p} \in S$ is given by

$$\boldsymbol{N}(\boldsymbol{p}) = \frac{1}{r}\,\boldsymbol{p}.$$

Thus, the Weingarten map is a scalar operator

$$-d_{\boldsymbol{p}}\boldsymbol{N}(\boldsymbol{w}) = -\frac{1}{r}\,\boldsymbol{w}.$$

In particular, the second fundamental form is

$$I\!I_p(\boldsymbol{w}) = \langle -d_p\boldsymbol{N}(\boldsymbol{w}), \boldsymbol{w}\rangle = -\frac{1}{r}\|\boldsymbol{w}\|^2.$$

Moreover, the eigenvalues are $\kappa_1(p) = \kappa_2(p) = -1/r$, the Gauss curvature is $K(p) = 1/r^2$ and the mean curvature is $H(p) = -1/r$.

Definition 9.7. Let S be a regular surface in $\mathbb{R}^3$ with Gauss map $\boldsymbol{N}\colon S \longrightarrow S^2$, and let $\boldsymbol{x}\colon U \longrightarrow S$ be a local parametrization. We call

$$L = \boldsymbol{x}_{uu}\cdot\boldsymbol{N}, \qquad M = \boldsymbol{x}_{uv}\cdot\boldsymbol{N} \quad \text{and} \quad N = \boldsymbol{x}_{vv}\cdot\boldsymbol{N}$$

the *coefficients of the second fundamental form.*

Proposition 9.8. *L, M, N are indeed the coefficients of $I\!I_p$ in the basis $\{\boldsymbol{x}_u, \boldsymbol{x}_v\}$, i.e.*

$$I\!I_p(a\boldsymbol{x}_u + b\boldsymbol{x}_v) = a^2L + 2abM + b^2N$$

Computing the matrix of the Weingarten map in the basis $\{\boldsymbol{x}_u, \boldsymbol{x}_v\}$ gives a matrix

$$-d_p\boldsymbol{N} = \frac{1}{EG - F^2}\begin{pmatrix} GL - FM & GM - FN \\ -FL + EM & -FM + EN \end{pmatrix},$$

which results in the following.

Proposition 9.9.

$$K = \frac{LN - M^2}{EG - F^2}, \qquad H = \frac{1}{2}\frac{EN - 2FM + GL}{EG - F^2}.$$

Example 9.10. Hyperbolic paraboloid.
Let $S := \{\,(x, y, z) \,|\, x^2 - y^2 + z = 0\,\}$. It may be parametrized as a graph of a function $z = f(x, y) = y^2 - x^2$, i.e., $\boldsymbol{x}(u, v) = (u, v, v^2 - u^2)$ for $(u, v) \in U = \mathbb{R}^2$. Then

$$\begin{aligned} \boldsymbol{x}_u &= (1, 0, -2u), & \boldsymbol{x}_v &= (0, 1, 2v), \\ \boldsymbol{x}_{uu} &= (0, 0, -2), \quad \boldsymbol{x}_{uv} = (0, 0, 0), & \boldsymbol{x}_{vv} &= (0, 0, 2). \end{aligned}$$

We also need the normal and calculate

$$\boldsymbol{x}_u \times \boldsymbol{x}_v = (2u, -2v, 1),$$

which has norm $D = (4u^2 + 4v^2 + 1)^{1/2}$, hence

$$\boldsymbol{N}\circ\boldsymbol{x} = \frac{1}{D}(2u, -2v, 1).$$

The coefficients of the 1^{st}FF and 2^{nd}FF are

$$E = \boldsymbol{x}_u \cdot \boldsymbol{x}_u = 1 + 4u^2, \quad F = \boldsymbol{x}_u \cdot \boldsymbol{x}_v = -4uv, \quad G = \boldsymbol{x}_v \cdot \boldsymbol{x}_v = 1 + 4v^2$$
$$L = \boldsymbol{x}_{uu} \cdot \boldsymbol{N} = \frac{-2}{D}, \quad M = \boldsymbol{x}_{uv} \cdot \boldsymbol{N} = 0, \quad N = \boldsymbol{x}_{vv} \cdot \boldsymbol{N} = \frac{2}{D}.$$

Now,

$$EG - F^2 = (1 + 4u^2)(1 + 4v^2) - 16u^2v^2 = 1 + 4u^2 + 4v^2 = D^2 \quad \text{and} \quad LN - M^2 = \frac{-4}{D^2},$$

so that the Gauss curvature is

$$K = \frac{LN - M^2}{EG - F^2} = \frac{-4}{D^4} < 0$$

and the mean curvature is

$$H = \frac{EN + GL}{2(EG - F^2)} = \frac{(1 + 4u^2) - (1 + 4v^2)}{D^3} = \frac{4(u^2 - v^2)}{D^3}.$$

Let us calculate the principal curvatures at $\boldsymbol{x}(0, 0) = (0, 0, 0)$ (i.e., $(u, v) = (0, 0)$). Here, $K = -4$ and $H = 0$, hence we look for the roots κ of

$$\kappa^2 - 2H\kappa + K = 0, \quad \text{or,} \quad \kappa^2 - 4 = 0,$$

i.e., $\kappa_1 = 2$ and $\kappa_2 = -2$.

Definition 9.11. A parametrization $\boldsymbol{x}$ with $F = 0$ is called *orthogonal*, a parametrization $\boldsymbol{x}$ with $F = 0$ and $M = 0$ is called *principal*.

Proposition 9.12. Assume that the parametrization $\boldsymbol{x}$ of a surface is principal (i.e., $F = 0$ and $M = 0$), then $\boldsymbol{x}_u$ and $\boldsymbol{x}_v$ are the principal directions. Moreover, the principal curvatures are

$$\kappa_1 = \frac{L}{E} \qquad \text{and} \qquad \kappa_2 = \frac{N}{G}.$$

Hence, the Gauss and mean curvatures are

$$K = \kappa_1\kappa_2 = \frac{LN}{EG} \qquad \text{and} \qquad H = \frac{1}{2}(\kappa_1 + \kappa_2) = \frac{GL + EN}{2EG}.$$

Example 9.13. Surface of revolution. Let S be obtained by rotating the curve given by $\boldsymbol{\alpha}(v) = (f(v), 0, g(v))$, $v \in I$ (some open interval) around the z-axis. Let us assume that $f(v) > 0$. A local parametrization is then given by

$$\boldsymbol{x}(u, v) = \begin{pmatrix} f(v)\cos u \\ f(v)\sin u \\ g(v) \end{pmatrix}$$

for $(u, v) \in U_1 = (0, 2\pi) \times I$ (and $(u, v) \in U_2 = (-\pi, \pi) \times I$ to cover the surface entirely). The derivatives are

$$\boldsymbol{x}_u = \begin{pmatrix} -f(v)\sin u \\ f(v)\cos u \\ 0 \end{pmatrix} \quad \text{and} \quad \boldsymbol{x}_v = \begin{pmatrix} f'(v)\cos u \\ f'(v)\sin u \\ g'(v) \end{pmatrix}.$$

For the coefficients of the second fundamental form, we also need the *second derivatives* of $\boldsymbol{x}$:

$$\boldsymbol{x}_{uu} = \begin{pmatrix} -f(v)\cos u \\ -f(v)\sin u \\ 0 \end{pmatrix}, \quad \boldsymbol{x}_{uv} = \boldsymbol{x}_{vu} = \begin{pmatrix} -f'(v)\sin u \\ f'(v)\cos u \\ 0 \end{pmatrix} \quad \text{and} \quad \boldsymbol{x}_{vv} = \begin{pmatrix} f''(v)\cos u \\ f''(v)\sin u \\ g''(v) \end{pmatrix}.$$

The normal vector at $p = \boldsymbol{x}(u, v)$ is

$$\boldsymbol{N}(p) = \Big(\frac{1}{\|\boldsymbol{x}_u \times \boldsymbol{x}_v\|}\boldsymbol{x}_u \times \boldsymbol{x}_v\Big)(u, v) = \frac{1}{\boldsymbol{\alpha}'(v)} \begin{pmatrix} g'(v)\cos u \\ g'(v)\sin u \\ -f'(v) \end{pmatrix},$$

where $\|\boldsymbol{\alpha}'(v)\| = (f'(v)^2 + g'(v)^2)^{1/2}$. Now, the coefficients of the second fundamental form are

$$L = \boldsymbol{x}_{uu} \cdot \boldsymbol{N} = \frac{-fg'}{\|\boldsymbol{\alpha}'\|}, \qquad M = \boldsymbol{x}_{uv} \cdot \boldsymbol{N} = 0 \quad \text{and}$$
$$N = \boldsymbol{x}_{vv} \cdot \boldsymbol{N} = \frac{f''g' - f'g''}{\|\boldsymbol{\alpha}'\|}.$$

The coefficients of the 1^{st}FF

$$E = \langle \boldsymbol{x}_u, \boldsymbol{x}_u \rangle = f^2, \quad F = \langle \boldsymbol{x}_u, \boldsymbol{x}_v \rangle = 0 \quad \text{and} \quad G = \langle \boldsymbol{x}_u, \boldsymbol{x}_v \rangle = \|\boldsymbol{\alpha}'\|^2.$$

Now we can calculate all the curvatures. The principal curvatures are

$$\kappa_1 = \frac{L}{E} = \frac{-fg'}{f^2\|\boldsymbol{\alpha}'\|} = \frac{-g'}{f\|\boldsymbol{\alpha}'\|} \qquad \text{and} \qquad \kappa_2 = \frac{N}{G} = \frac{f''g' - f'g''}{\|\boldsymbol{\alpha}'\|^3}.$$

Hence, the Gauss and mean curvatures are

$$K = \kappa_1\kappa_2 = \frac{LN}{EG} = \frac{-g'(f''g' - f'g'')}{f\|\boldsymbol{\alpha}'\|^4} \quad \text{and}$$
$$H = \frac{1}{2}(\kappa_1 + \kappa_2) = \frac{-g'}{2f} + \frac{f''g' - f'g''}{2\|\boldsymbol{\alpha}'\|^3}.$$

Example 9.14. Torus of revolution. Apply the above to the case $f(v) = R + r\cos(v/r)$ and $g(v) = r\sin(v/r)$, $0 < r < R$. Calculate the principal, Gauss curvature and mean curvatures.

We just calculate

$$f'(v) = -\sin(v/r), \qquad g'(v) = \cos(v/r),$$
$$f''(v) = -\frac{1}{r}\cos(v/r), \qquad g''(v) = -\frac{1}{r}\sin(v/r).$$

so that

$$\kappa_1 = \frac{-g'}{f} = \frac{\cos}{R + r\cos} \qquad \text{and} \qquad \kappa_2 = \frac{f''g' - f'g''}{f} = -\frac{1}{r}(\cos^2 + \sin^2) = -\frac{1}{r}$$

since $(f')^2 + (g')^2 = 1$ (the arguments of cos and sin in this formula are v/r). In particular, one principal curvature is constant (it is the one coming from going around the torus along the small circle, i.e., in direction $\boldsymbol{x}_u$). Moreover,

$$K = \kappa_1\kappa_2 = \frac{\cos}{r(R + r\cos)} \qquad \text{and} \qquad H = \frac{\cos}{2(R + r\cos)} - \frac{1}{2r} = \frac{-R}{2r(R + r\cos)}.$$

Note that the mean curvature never vanishes.

Definition 9.15.

(a) Let S be a surface and $K(p)$ its Gauss curvature at $p \in S$. We say that p is

$$\begin{cases} \textit{elliptic} & & K(p) > 0 \\ \textit{hyperbolic} & \text{if} & K(p) < 0 \\ \textit{flat} & & K(p) = 0 \end{cases}$$

$$\text{The subset} \qquad \begin{cases} \{\, p \in S \mid K(p) > 0 \,\} \\ \{\, p \in S \mid K(p) < 0 \,\} \\ \{\, p \in S \mid K(p) = 0 \,\} \end{cases} \quad \text{is called} \quad \begin{matrix} \textit{elliptic} \\ \textit{hyperbolic} \\ \textit{flat} \end{matrix} \quad \text{region of } S$$

(b) Denote by $\kappa_1(p)$ and $\kappa_2(p)$ the principal curvatures at $p \in S$.

- We say that p is *planar* if $\kappa_1(p) = 0$ and $\kappa_2(p) = 0$;
- we say that p is *umbilic* if $\kappa_1(p) = \kappa_2(p)$.

Example 9.16. (a) (Sphere) On a sphere $S^2(r)$, all points are elliptic and umbilic since both principal curvatures are $\kappa_1(p) = \kappa_2(p) = -1/r$. The converse is also true (see Theorem 9.19).

(b) (Plane) It is not hard to see that if S is a plane (or an open subset of it) then all points of S are planar. The converse is also true (see Theorem 9.19).

(c) (Hyperbolic paraboloid, Example 9.10) All points are hyperbolic (since $K(p) < 0$ for all $p \in S$), and in particular, there are no umbilic points or flat points.

(d) (Torus of revolution, Example 9.14) We have $K = 0$ iff $\cos(v/r) = 0$ i.e., if $v/r = \pi/2$ or $v/r = 3\pi/2$. This is the circle on top and bottom of the torus; this is the *flat region*. The *elliptic region* is given by points with $K > 0$, i.e., $-\pi/2 < v/r < -\pi/2$. The *hyperbolic region* is given by points with $K < 0$, i.e., $\pi/2 < v/r < 3\pi/2$.

There are no umbilic points on the torus of revolution: $|\kappa_1| < 1/r$, but $\kappa_2 = -1/r$, so the two principal curvatures cannot be the same. There are no planar points either $\kappa_2 = -1/r \neq 0$ everywhere).

9.2 Some global theorems about curvature

Theorem 9.17. Every compact surface in $\mathbb{R}^3$ has at least one elliptic point.

Remark 9.18. The theorem is obviously false if either boundedness or closedness is dropped.

Theorem 9.19. Let S be a surface in $\mathbb{R}^3$.

(a) If all points of S are umbilic and $K \neq 0$ in at least one point of S then S is a part of a sphere.

(b) If all points of S are planar then S is part of a plane.

Theorem 9.20 (Conjecture of Carathéodory)**.** Every compact surface in $\mathbb{R}^3$ (convex, homeomorphic to a sphere) has at least two umbilic points.

This theorem has recently (2008) been proved (with additional smoothness assumptions) by Brendan Guilfoyle and Wilhelm Klingenberg (Durham).

Definition 9.21. A surface S is *minimal* if the mean curvature H vanishes identically on S.