Durham University
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Analysis III/IV, Solutions 1 (Weeks 1–2)

Starred problems are **more difficult** and are **not for submission**.

Real numbers. Countable and uncountable sets.

1.1. Show that there exist $n, k \in \mathbb{N}$ such that

(a) $(1+\frac{1}{1000000})^n > 1000000$;
(b) $(1-\frac{1}{10000})^k < \frac{1}{1000000}$.

Solution:

(a) For every positive $a \in \mathbb{R}$ and for any $n \in \mathbb{N}$ one has $(1+a)^n \geq 1+na$. By the Archimedean property of $\mathbb{N}$, for any positive $a, b \in \mathbb{R}$ there exists $n \in \mathbb{N}$ such that $na > b$. Thus, taking $a = \frac{1}{1000000}$ and $b = 1000000$, we see that there exists $n \in \mathbb{N}$ such that

$$(1+\frac{1}{1000000})^n \geq 1+\frac{1}{1000000}n > 1000000.$$

(b) We reduce this to the previous exercise (though it could be solved directly as well). Observe that $1-\frac{1}{10000} < 1-\frac{1}{1000000}$, which implies $(1-\frac{1}{10000})^k < (1-\frac{1}{1000000})^k$. Now observe that $(1-\frac{1}{1000000})(1+\frac{1}{1000000}) < 1$, so $(1-\frac{1}{1000000})^k(1+\frac{1}{1000000})^k < 1$. Therefore, for $k = n$ from (a), we obtain

$$(1-\frac{1}{10000})^k < (1-\frac{1}{1000000})^k < \frac{1}{(1+\frac{1}{1000000})^k} < \frac{1}{1000000}.$$

1.2. Without using uncountability of $\mathbb{R}$, show that $\mathbb{R} \setminus \mathbb{Q}$ is not empty.
Hint: consider the set $\{x \in \mathbb{R} \,|\, x^2 < 3\}$.

Solution:

Let $S = \{x \in \mathbb{R} \,|\, x^2 < 3\}$. S is bounded from above (for example, by 2), so there exists $s = \sup S$. We want to prove two statements: first, that $s^2 = 3$ (call it Claim 1), and second, that $s \notin \mathbb{Q}$ (Claim 2). This will complete the proof.

Proof of Claim 1.
Suppose that $s^2 < 3$, i.e. $s^2 = 3-\epsilon$, $\epsilon > 0$. Let $\delta \in (0,1)$. Then

$$(s+\delta)^2 = s^2+2s\delta+\delta^2 = 3-\epsilon+2s\delta+\delta^2 < 3-\epsilon+2s\delta+\delta = 3-\epsilon+\delta(2s+1).$$

Now, taking δ small enough, we can assume that $\delta(2s+1) < \epsilon$ (deduce this from the Archimedean property of $\mathbb{R}$). Thus, there exists $\delta > 0$, such that $(s+\delta)^2 < 3$, which contradicts s being the supremum of S.

In a similar way, one can prove that an assumption $s^2 > 3$ leads to a contradiction (do it!). Thus, $s^2 = 3$.

Proof of Claim 2.
Now, suppose $s = p/q \in \mathbb{Q}$, where $p \in \mathbb{Z}$ and $q \in \mathbb{N}$. We can also assume p and q have no common divisors. Then $p^2 = 3q^2$. This implies that p is divisible by 3, i.e. $p = 3\tilde{p}$, $\tilde{p} \in \mathbb{Z}$. Therefore, $3\tilde{p}^2 = q^2$, so q is divisible by 3, which contradicts to the assumtion of p and q being coprime.

1.3. Is it true that

(a) If $|A| = |B|$ and $|C| = |D|$, then $|A \times C| = |B \times D|$?
(b) If $|A| = |B|$ and $|C| = |D|$, then $|A \cup C| = |B \cup D|$?
(c) An interval (a, b) (where $a < b$) is equipotent to $\mathbb{R}$?

Solution:

(a) Let $f: A \to B$ and $g: C \to D$ be bijections. Then the map $h: A \times C \to B \times D$ defined by

$$h(a, c) = (f(a), g(c))$$

is a bijection, so the statement is always true.

(b) Counterexample:

$$A = B = C = \{1\}, \quad D = \{2\}.$$

Then $A \cup C = \{1\}$, $B \cup D = \{1, 2\}$, so they are not equipotent. Therefore, the statement may not be true for sets satisfying the assumptions.
Note that if we add an assumption that $A \cap C = B \cap D = \emptyset$, then the statement will become true (prove it!).

(c) One way is to write the formula explicitly: take $f : (a, b) \to \mathbb{R}$,

$$f(x) = \tan \frac{\pi}{b-a} \left(x - \frac{b+a}{2} \right)$$

(note that the formula also tells us that any two intervals are equipotent).
Alternatively, this can be done geometrically: an interval I is equipotent to a semi-circle without endpoints with dimater I (via orthogonal projection to the interval), and a semi-circle is equipotent to a line via a projection from the center.

1.4. Show that

(a) Every infinite set has a countable infinite subset.
(b) If A is countable and B is infinite, then $|A \cup B| = |B|$.
(c) If A is countable and B is uncountable, then $|B \setminus A| = |B|$.

Solution:

(a) Let B be infinite, take $x_1 \in B$. Then take $x_2 \in B_1 = B \setminus \{x_1\}$, and in general $x_{k+1} \in B_k = B \setminus \cup_{i \le k} x_i$. Since B is infinite, every B_k is not empty. Now $C = \cup_{i \in \mathbb{N}} \{x_i\}$ $(= B \setminus (\cap_{i \in \mathbb{N}} B_i))$ is countable, $C \subseteq B$.

(b) By (a), we can take a countable infinite $C \subseteq B$. Then $B = C \cup (B \setminus C)$, and $A \cup B = (A \cup C) \cup (B \setminus C) = ((A \setminus B) \cup C) \cup (B \setminus C)$. Now, $A \setminus B$ is countable as a subset of a countable set, so $((A \setminus B) \cup C)$ is also countable infinite as a union of two countable sets (at least one of which is infinite). Therefore, $|((A \setminus B) \cup C)| = |C|$, and since both sets do not intersect with $(B \setminus C)$, we have (see the solution of Exercise 1.3(b))

$$|A \cup B| = |((A \setminus B) \cup C) \cup (B \setminus C)| = |C \cup (B \setminus C)| = |B|.$$

More precisely, the bijective map from $A \cup B$ to B is constructed in the following way: it is an identity on $B \setminus C$, and maps elements of $(A \setminus B) \cup C$ to elements of C according to a bijection between two infinite countable sets.

(c) The problem is similar to the previous one. Choose a countable infinite $C \subseteq (B \setminus A)$, then $C \cup (A \cap B)$ is countable infinite as a union of two countable sets (at least one of which is infinite). Therefore,

$$|B \setminus A| = |(B \setminus (A \cup C)) \cup C| = |(B \setminus (A \cup C)) \cup ((A \cap B) \cup C)| = |B|.$$

Alternatively, we can reduce the problem to (b): $B \setminus A$ is uncountable and thus infinite, $A \cap B$ is countable, so $|B \setminus A| = |(B \setminus A) \cup (A \cap B)| = |B|$.

1.5. (Power Set)
Recall that the *power set* $P(A)$ of a set A is the set of all subsets of A, i.e. $P(A) = \{S \,|\, S \subseteq A\}$.

(a) Show that if $|A| = n \in \mathbb{N}$, then $|P(A)| = 2^n$.
(b) Show that $P(A)$ is not equipotent to A for any set A.
Hint: suppose that $f : A \to P(A)$ is a bijection, and consider the set $\{a \in A \,|\, a \in f(a)\} \subseteq A$.

Solution:

(a) Let $A = \{x_1, \ldots, x_n\}$. We can assign to every subset $S \subseteq A$ a sequence of zeroes and ones: we write zero on k-th place if $x_k \notin S$, and 1 otherwise. It is easy to see that this is a bijection of $P(A)$ onto the set of sequences of length n, which has 2^n elements.

(b) Following the hint, suppose that $f : A \to P(A)$ is a bijection, and consider the set $S = \{a \in A \,|\, a \in f(a)\} \subseteq A$. Since f is a bijection, there exists $s \in A$ such that $f(s) = A \setminus S = \{a \in A \,|\, a \notin f(a)\} \subseteq A$.
If $s \in A \setminus S$, then $s \notin f(s) = A \setminus S$ by the definition of S, so we have a contradiction. Similarly, if $s \notin A \setminus S$, then $s \in f(s) = A \setminus S$ by the definition of S, so we have a contradiction again. This implies that a bijection cannot exist, so the sets A and $P(A)$ are not equipotent.

1.6. ($\star$) **(Cantor – Bernstein Theorem)**
Given two sets A and B, we write $|A| \le |B|$ if there exists an injective map $A \to B$. Show that if both $|A| \le |B|$ and $|B| \le |A|$ hold, then A and B are equipotent.