Durham University
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Analysis III/IV, Homework 1 (Weeks 1–2)

Due date: Thursday, October 26.

Starred problems are **more difficult** and are **not for submission**.

Real numbers. Countable and uncountable sets.

1.1. Show that there exist $n, k \in \mathbb{N}$ such that

(a) $(1 + \frac{1}{1000000})^n > 1000000$;
(b) $(1 - \frac{1}{10000})^k < \frac{1}{1000000}$.

1.2. Without using uncountability of $\mathbb{R}$, show that $\mathbb{R} \setminus \mathbb{Q}$ is not empty.
Hint: consider the set $\{x \in \mathbb{R} \mid x^2 < 3\}$.

1.3. Is it true that

(a) If $|A| = |B|$ and $|C| = |D|$, then $|A \times C| = |B \times D|$?
(b) If $|A| = |B|$ and $|C| = |D|$, then $|A \cup C| = |B \cup D|$?
(c) An interval (a, b) (where $a < b$) is equipotent to $\mathbb{R}$?

1.4. Show that

(a) Every infinite set has a countable infinite subset.
(b) If A is countable and B is infinite, then $|A \cup B| = |B|$.
(c) If A is countable and B is uncountable, then $|B \setminus A| = |B|$.

1.5. (Power Set)
Recall that the *power set* $P(A)$ of a set A is the set of all subsets of A, i.e. $P(A) = \{S \mid S \subseteq A\}$.

(a) Show that if $|A| = n \in \mathbb{N}$, then $|P(A)| = 2^n$.
(b) Show that $P(A)$ is not equipotent to A for any set A.
Hint: suppose that $f : A \to P(A)$ is a bijection, and consider the set $\{a \in A \mid a \in f(a)\} \subseteq A$.

1.6. ($\star$) **(Cantor – Bernstein Theorem)**
Given two sets A and B, we write $|A| \leq |B|$ if there exists an injective map $A \to B$. Show that if both $|A| \leq |B|$ and $|B| \leq |A|$ hold, then A and B are equipotent.