Durham University
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Analysis III/IV, Solutions 2 (Weeks 3–4)

Starred problems are **more difficult** and are **not for submission**.

Open and closed sets. Sequences in $\mathbb{R}$.

2.1. Let $A \subseteq \mathbb{R}$. A point $x \in \mathbb{R}$ is called an *accumulation point of* A if x is a closure point of $A \setminus \{x\}$. We denote by A' the set of all accumulation points of A. Show that

(a) the set A' is closed;

(b) $\bar{A} = A \cup A'$;

(c) if A is infinite, closed and bounded, then A' is not empty.

Solution:

(a) Take any point of closure x of A', we need to prove that $x \in A'$. Let $\varepsilon > 0$, denote $I_x = (a - \varepsilon, a + \varepsilon)$. Then there exists $y \in A' \cap I_x$. If $y = x$ then $x \in A'$ and we are done, so assume $y \neq x$. Since $y \in A'$, for every $\varepsilon' > 0$ there exists $a \in A \cap (y - \varepsilon', y + \varepsilon')$, $a \neq y$. Take ε' small enough such that $(y - \varepsilon', y + \varepsilon') \subset I_x \setminus \{x\}$. Therefore, we proved that for every point of closure x of A' and for every $\varepsilon > 0$ there exists $a \in A \cap I_x \setminus \{x\}$, which means that $x \in A'$, so $\bar{A'} \subset A'$, and thus A' is closed.

(b) First, both A and A' are subsets of $\bar{A}$ by definition, so $A \cup A' \subseteq \bar{A}$. Conversely, let $x \notin A'$. Then there exists $\varepsilon > 0$ such that $(x - \varepsilon, x + \varepsilon) \cap (A \setminus \{x\}) = \emptyset$, so $x \in \bar{A} \setminus A'$ implies $x \in A$. Thus, is x is a closure point of A, then $x \in A \cup A'$, so $\bar{A} \subseteq A \cup A'$.

(c) Since A is infinite, there is an infinite countable subset of A, which gives rise to a sequence with all elements being distinct. By Bolzano – Weierstrass theorem, there is an accumulation point a of this sequence. Since all the elements are distinct, a is also an accumulation point of A (note that the closedeness of A is actually not required).

2.2. Let $A \subseteq \mathbb{R}$. A point $x \in \mathbb{R}$ is called an *isolated point of* A if there exists an open interval $I_x \ni x$ not containing any other point of A. Show that

(a) if $x \in A$, then either $x \in A'$ or x is an isolated point of A;

(b) If every point of A is isolated, then A is countable.

Solution:

(a) If $a \notin A'$, then, as we have seen above, there exists $\varepsilon > 0$ such that $(a - \varepsilon, a + \varepsilon)$ contains no points of A except a, which is the definition of an isolated point.

(b) For every $a \in A$ there exists $\varepsilon_a > 0$ such that $(a - \varepsilon_a, a + \varepsilon_a)$ contains no points of A except a. Denote $I_a = (a - \varepsilon_a/2, a + \varepsilon_a/2)$. Then for any two points $a_1, a_2 \in A$ the intervals I_{a_1} and I_{a_2} are disjoint. Taking a rational number $q_a \in I_a$ for every $a \in A$, we construct a bijection between A and a subset of $\mathbb{Q}$, so A is countable.

2.3. A set $A \subseteq \mathbb{R}$ is called *perfect* if $A = A'$. Show that A is perfect if and only if $A \subseteq A'$ and A is closed.

Solution:

Let A be closed, and assume $A \subseteq A'$. Then $A' = A' \cup A = \bar{A} = A$ (see Exercise 2.1(b)), so A is perfect.

Conversely, if A is perfect, then $A = A'$, so $A \subseteq A'$, and A is closed since A' is always closed (see Exercise 2.1(a)).

2.4. ($\star$) Does there exist a countable perfect set?

2.5. Let $a \in \mathbb{R}$ or $a = \pm\infty$, and let $\{x_n\}$ be a sequence of real numbers. Show that $a = \lim_{n\to\infty} x_n$ if and only if a is the only accumulation point of $\{x_n\}$.

Solution:

Consider the case $a \in \mathbb{R}$.

Assume first that $a = \lim_{n\to\infty} x_n$. Suppose there is another accumulation point b, take $\varepsilon = |b-a|/2$. Since a is the limit, there is $N \in \mathbb{N}$ such that $|x_n - a| < \varepsilon$ for every $n > N$. This means that no x_n with $n > N$ satisfies $|x_n - b| < \varepsilon$, so b is not an accumulation point.

Conversely, suppose a is the only accumulation point of $\{x_n\}$ but not the limit, so there exists $\varepsilon > 0$ such that infinitely many elements of $\{x_n\}$ lie outside the interval $(a-\varepsilon, a+\varepsilon)$, choose a subsequence $\{x_{n_k}\}$ satisfying $|x_{n_k} - a| \geq \varepsilon$. If $\{x_{n_k}\}$ is bounded, then (by Bolzano – Weierstrass theorem) it has an accumulation point distinct from a, so we come to a contradiction. If $\{x_{n_k}\}$ is unbounded, then $\pm\infty$ is an accumulation point, so we have a contradiction again.

Now suppose $a = \pm\infty$, say $a = \infty$.

$\lim_{n\to\infty} x_n = \infty$ means that for every $C \in \mathbb{R}$ there exists $N \in \mathbb{N}$ such that $x_n > C$ for every $n > N$, which is equivalent to the following statement: for every $C \in \mathbb{R}$ there are finitely many elements of $\{x_n\}$ which are smaller than C, and infinitely many elements which are larger that C. This, in its turn, is equivalent to the fact that ∞ is an accumulation point, and any other element of $\mathbb{R} \cup \pm\infty$ is not.

2.6. (Base-p expansions)
Let $p > 1$ be a natural number, and let $x \in (0,1)$.

(a) Show that there exists a sequence $\{a_n\}$ of integers such that $0 \leq a_n < p$ for every $n \in \mathbb{N}$, and

$$x = \sum_{n=1}^{\infty} \frac{a_n}{p^n}.$$

(b) Show that the sequence $\{a_n\}$ in (a) is unique unless $x = q/p^n$ for some $q \in \mathbb{N}$, in which case there are precisely two such sequences.

(c) Show that for every sequence $\{a_n\}$ of integers satisfying $0 \leq a_n < p$ for every $n \in \mathbb{N}$, the series

$$\sum_{n=1}^{\infty} \frac{a_n}{p^n}$$

converges to some $x \in [0,1]$.

Solution:

(a) Given $x \in \mathbb{R}$, let a_1 be the largest integer such that $0 \leq a_1 < p$ and $a_1/p \leq x$. Suppose $a_1, \ldots, a_n$ have been chosen. Let a_{n+1} be the largest integer such that $0 \leq a_{n+1} < p$ and $\frac{a_{n+1}}{p^{n+1}} \leq x - \sum_{k=1}^{n} \frac{a_k}{p^k}$. This gives rise to a sequence $\{a_n\}$ of integers with $0 \leq a_n < p$ and $x - \sum_{k=1}^{n} \frac{a_k}{p^k} < 1/p^n$ for all $n \in \mathbb{N}$. Now, given $\varepsilon > 0$, there exists N such that $1/p^N < \varepsilon$. Then $x - \sum_{k=1}^{n} \frac{a_k}{p^k} < 1/p^N < \varepsilon$ for all $n > N$, and therefore $x = \sum_{k=1}^{\infty} \frac{a_k}{p^k}$.

(b) Let $x \in (0,1)$, and suppose there are two distinct sequences $\{a_n\}$ and $\{b_n\}$ such that $\sum_{k=1}^{\infty} \frac{b_k}{p^k} = x = \sum_{k=1}^{\infty} \frac{a_k}{p^k}$. Let m be the smallest index for which $b_m \neq a_m$, assume without loss of generality that $a_m < b_m$. Observe that

$$0 = \sum_{k=1}^{\infty} \frac{b_k}{p^k} - \sum_{k=1}^{\infty} \frac{a_k}{p^k} = \sum_{k=m}^{\infty} \frac{b_k}{p^k} - \sum_{k=m}^{\infty} \frac{a_k}{p^k} = \frac{b_m - a_m}{p^m} + \left(\sum_{k=m+1}^{\infty} \frac{b_k}{p^k} - \sum_{k=m+1}^{\infty} \frac{a_k}{p^k} \right) =$$
$$\frac{b_m - a_m}{p^m} - \left(\sum_{k=m+1}^{\infty} \frac{a_k}{p^k} - \sum_{k=m+1}^{\infty} \frac{b_k}{p^k} \right) \geq \frac{1}{p^m} - \left(\sum_{k=m+1}^{\infty} \frac{p-1}{p^k} - \sum_{k=m+1}^{\infty} \frac{0}{p^k} \right) = \frac{1}{p^m} - \sum_{k=m+1}^{\infty} \frac{p-1}{p^k} = 0.$$

Therefore, to have an equality in the inequality above, it is necessary and sufficient to have $a_m = b_m - 1$, and for all $k > m$ every $b_k = 0$ and every $a_k = p - 1$, which produces precisely two distinct sequences. This is equivalent to $x = \frac{q}{p^m}$, where $q = \sum_{k=1}^{m} b_k p^{m-k}$.

Remark.

Geometrically, the proofs of (a) and (b) above can be interpreted in the following way. We subdivide the interval $(0, 1)$ into p equal parts (indexed from 0 to $p - 1$), and a_1 is the number of the part where x belongs to. We then subdivide this part into p equal parts, and now a_2 is the number of the part where x belongs to, etc. Taking $a_m < b_m$ we put the point in different segments of length $1/p^m$, so the sequences give rise to the same number in the only case when this number is the intersection of these two segments. This, in its turn, can happen in the only case when the segments are neighboring (i.e., $b_m = a_m + 1$), and x is the left endpoint of the right segment (i.e., all $b_k = 0$ for $k > m$) and the right endpoint of the left segment (i.e., all $a_k = p - 1$ for $k > m$).

(c) Let $\{a_n\}$ be a sequence of integers with $0 \le a_n < p$, let $s_n = \sum_{k=1}^{n} \frac{a_k}{p^k}$. Then $0 \le s_n \le (p-1)\sum_{k=1}^{\infty} \frac{1}{p^k} = 1$. Thus, $\{s_n\}$ is a bounded monotone increasing sequence, so it converges. Furthermore, since $0 \le s_n \le 1$ for all $n \in \mathbb{N}$, the sequence converges to a real number $x \in [0, 1]$.

2.7. ($\star$) **(Continued fractions)**

Let $\{a_n\}$ be any sequence of natural numbers. Define a sequence $\{x_n\}$ by

$$x_1 = a_1, \quad x_2 = a_1 + \frac{1}{a_2}, \quad x_3 = a_1 + \frac{1}{a_2 + \frac{1}{a_3}}, \quad x_4 = a_1 + \frac{1}{a_2 + \frac{1}{a_3 + \frac{1}{a_4}}}, \quad \ldots$$

(a) Show that $\{x_n\}$ converges to some $x \in \mathbb{R}$.

(b) Find $\lim_{n\to\infty} x_n$ for $a_n = 1\,\forall n \in \mathbb{N}$.

(c) Find $\lim_{n\to\infty} x_n$ if $\forall n \in \mathbb{N}$ $a_{3n-2} = 1$, $a_{3n-1} = 2$, $a_{3n} = 3$.

(d) Find $\{a_n\}$ such that $\lim_{n\to\infty} x_n = \sqrt{7}$.