Durham University
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Analysis III/IV, Homework 2 (Weeks 3–4)

Due date: Thursday, November 9.

Starred problems are **more difficult** and are **not for submission**.

Open and closed sets. Sequences in $\mathbb{R}$.

2.1. Let $A \subseteq \mathbb{R}$. A point $x \in \mathbb{R}$ is called an *accumulation point of A* if x is a closure point of $A \setminus \{x\}$. We denote by A' the set of all accumulation points of A. Show that

(a) the set A' is closed;
(b) $\bar{A} = A \cup A'$;
(c) if A is infinite, closed and bounded, then A' is not empty.

2.2. Let $A \subseteq \mathbb{R}$. A point $x \in \mathbb{R}$ is called an *isolated point of A* if there exists an open interval $I_x \ni x$ not containing any other point of A. Show that

(a) if $x \in A$, then either $x \in A'$ or x is an isolated point of A;
(b) If every point of A is isolated, then A is countable.

2.3. A set $A \subseteq \mathbb{R}$ is called *perfect* if $A = A'$. Show that A is perfect if and only if $A \subseteq A'$ and A is closed.

2.4. ($\star$) Does there exist a countable perfect set?

2.5. Let $a \in \mathbb{R}$ or $a = \pm\infty$, and let $\{x_n\}$ be a sequence of real numbers. Show that $a = \lim\limits_{n\to\infty} x_n$ if and only if a is the only accumulation point of $\{x_n\}$.

2.6. (Base-p expansions)
Let $p > 1$ be a natural number, and let $x \in (0, 1)$.

(a) Show that there exists a sequence $\{a_n\}$ of integers such that $0 \leq a_n < p$ for every $n \in \mathbb{N}$, and

$$x = \sum_{n=1}^{\infty} \frac{a_n}{p^n}.$$

(b) Show that the sequence $\{a_n\}$ in (a) is unique unless $x = q/p^n$ for some $q \in \mathbb{N}$, in which case there are precisely two such sequences.

(c) Show that for every sequence $\{a_n\}$ of integers satisfying $0 \leq a_n < p$ for every $n \in \mathbb{N}$, the series

$$\sum_{n=1}^{\infty} \frac{a_n}{p^n}$$

converges to some $x \in [0, 1]$.

2.7. ($\star$) **(Continued fractions)**
Let $\{a_n\}$ be any sequence of natural numbers. Define a sequence $\{x_n\}$ by

$$x_1 = a_1, \quad x_2 = a_1 + \frac{1}{a_2}, \quad x_3 = a_1 + \frac{1}{a_2 + \frac{1}{a_3}}, \quad x_4 = a_1 + \frac{1}{a_2 + \frac{1}{a_3 + \frac{1}{a_4}}}, \quad \ldots$$

(a) Show that $\{x_n\}$ converges to some $x \in \mathbb{R}$.
(b) Find $\lim\limits_{n\to\infty} x_n$ for $a_n = 1\, \forall n \in \mathbb{N}$.
(c) Find $\lim\limits_{n\to\infty} x_n$ if $\forall n \in \mathbb{N}$ $a_{3n-2} = 1$, $a_{3n-1} = 2$, $a_{3n} = 3$.
(d) Find $\{a_n\}$ such that $\lim\limits_{n\to\infty} x_n = \sqrt{7}$.